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Invariant Subspaces of Bergman Spaces on Slit Domains

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INVARIANT SUBSPACES OF BERGMAN SPACES ON SLIT DOMAINS

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Abstract. In this paper, we characterize the z-invariant subspaces that lie between the Bergman spaces $A^p(G)$ and $A^p(G\backslash K)$, where $1 < p < \infty$, G is a bounded region in $\mathbb{C}$, and K is a closed subset of a simple, compact, C^1 arc.

1. Introduction

For a bounded region $U \subset \mathbb{C}$ and $1 < p < \infty$, we define the *Bergman space* $A^p(U)$ to be the space of analytic functions f on U with $\int_U |f(z)|^p dA(z) < \infty$ (here dA is Lebesgue measure on $\mathbb{C}$), and the operator S on $A^p(U)$ by

$$(Sf)(z) = zf(z).$$

Characterizing the S-invariant subspaces of $A^p(U)$ (those subspaces $\mathcal{M}$ of $A^p(U)$ with $S\mathcal{M} \subset \mathcal{M}$) is a difficult and unsolved problem.

This paper investigates the S-invariant subspaces $\mathcal{M}$ of $A^p(G\backslash K)$ (where G is a bounded region in the plane and K is a closed subset of a simple compact arc of class C^1) such that

$$A^p(G) \subset \mathcal{M} \subset A^p(G\backslash K). \tag{1.1}$$

(Throught this paper, G will be a bounded region, K will be the C^1 slit, and $\mathcal{M}$ will be an S-invariant subspace of the form (1.1).) The character and complexity of these invariant subspaces depends on the index p and splits into two distinct cases: $1 < p < 2$ and $p \geq 2$. For $1 < p < 2$, $\mathcal{M}$ can be described in terms of analytic continuation across portions of the 'slit' K.

Theorem 1.1. *For $1 < p < 2$, there is a closed set $F \subset K$ with $\mathcal{M} = A^p(G\backslash F)$.*

To prove Theorem 1.1, we use a technique of [2] [3] [6] to relate the annihilator of $A^p(U)$ with the Sobolev space $W_1^{q,0}(U)$, $q = p(p-1)^{-1}$, via the differential operator $\overline{\partial}$. This converts characterizing the z-invariant subspaces in (1.1) to the problem of characterizing the z-invariant subspaces that lie between the Sobolev spaces $W_1^{q,0}(G\backslash K)$ and $W_1^{q,0}(G)$. For $1 < p < 2$, the conjugate index $q > 2$, making $W_1^{q,0}(G)$ a Banach algebra of continuous functions. We then utilize the fact that $G\backslash K$ is a 'slit' domain to show that such subspaces are in fact closed ideals of $W_1^{q,0}(G)$ which, using Banach algebra techniques of Sarason [12], can be written as $W_1^{q,0}(G\backslash F)$. We then reverse the above process to obtain $\mathcal{M} = A^p(G\backslash F)$.

For $p \geq 2$, the situation becomes more complicated as not every $\mathcal{M}$ can be written as $A^p(G\backslash F)$, but we still can describe these subspaces by means of the q-capacity, C_q, associated with the Sobolev space W_1^q [2]. To state our second theorem, we make the following definition: We say a set $E \subset \mathbb{C}$ is *quasi-closed* if given any $\varepsilon > 0$, there is an open set W with $C_q(W) < \varepsilon$ and $E\backslash W$ closed.

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Theorem 1.2. *For $p \geq 2$, there is a quasi-closed set $E \subset K$ and an increasing sequence of closed sets $F_1 \subset F_2 \subset \cdots \subset E$ with $C_q(F_n) \to C_q(E)$ and*

$$\mathcal{M} = \mathcal{M}(E) = \overline{\bigcup_{n\geq 1} A^p(G\backslash F_n)}^{L^p}.$$

Moreover, $\mathcal{M}(E)$ is independent of the choice of $\{F_n\}$ and if $E_1, E_2 \subset K$ are quasi-closed, then $\mathcal{M}(E_1) = \mathcal{M}(E_2)$ if and only if $C_q(E_1 \Delta E_2) = 0$.

The characterization of the z-invariant subspaces between $W_1^{q,0}(G\backslash K)$ and $W_1^{q,0}(G)$ is at the heart of our problem and will use weak topology techniques of [9].

2. Preliminaries

For $1 < q < \infty$, define the *Sobolev space* $W_1^q = W_1^q(\mathbb{C})$ as the space of functions $u \in L^q = L^q(\mathbb{C}, dA)$ whose first partial derivatives (in the sense of distributions) are also in L^q. We norm W_1^q by

$$\|u\|_q = \Big(\int (|u|^2 + |\nabla u|^2)^{q/2} dA \Big)^{1/q}$$

and note that W_1^q is a separable, reflexive Banach space [1], Theorem 3.2 and Theorem 3.5.

For a bounded domain $U \subset \mathbb{C}$, define $W_1^{q,0}(U)$ to be the closure of $C_0^\infty(U)$ in the W_1^q norm and note, by the Poincaré inequality [8], p. 69, we can equivalently norm $W_1^{q,0}(U)$ by

$$\|u\|_{q,0} = (\int_U |\nabla u|^q dA)^{1/q}.$$

If $q > 2$, the Sobolev imbedding theorem yields $W_1^{q,0}(U)$ is a Banach algebra of continuous functions [1], p. 115.

We identify the dual of $L^p(U)$ with $L^q(U)$, $q = p(p-1)^{-1}$, via the bilinear pairing

$$< f, g >= \int_U fg dA, \quad f \in L^p(U), \quad g \in L^q(U)$$

and write $B^q(U)$ for the annihilator of $A^p(U)$ in $L^q(U)$. In general, we denote the annihilator of a set X by $X^\perp$. In [2] [3] they identify $B^q(U)$ with $W_1^{q,0}(U)$ via the operator

$$\overline{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

as follows: By Weyl's lemma [15], p. 32, $A^p(U) = (\overline{\partial} C_0^\infty(U))^\perp$. Now apply the Hahn-Banach theorem, to obtain:

Lemma 2.1. *$B^q(U)$ is the L^q-closure of $\overline{\partial} C_0^\infty(U)$.*

An application of the Calderon-Zygmund theory [14], p. 60, yields the equivalence of $\|\overline{\partial} v\|_{L^q}$ and $\|\nabla v\|_{L^q}$ for all $v \in C_0^\infty(U)$, i.e. there is a positive constant A_q depending only on q with

$$A_q \|\overline{\partial} v\|_{L^q} \leq \|\nabla v\|_{L^q} \leq A_q^{-1} \|\overline{\partial} v\|_{L^q} \quad \forall v \in C_0^\infty(U). \tag{2.1}$$

Define D densely on $W_1^{q,0}(U)$ by $Dv = \overline{\partial} v$, where $v \in C_0^\infty(U)$ and notice that

$$z\overline{\partial} v = \overline{\partial}(zv) \quad \forall v \in C_0^\infty(U). \tag{2.2}$$

By our bilinear pairing $< \cdot, \cdot >$, the operator R on $B^q(U)$ defined by $(Rg)(z) = zg(z)$ is well defined and continuous. If we define M on $W_1^{q,0}(U)$ by $(Mh)(z) = zh(z)$ and use the above lemma, (2.1), and (2.2), we have:

Proposition 2.2. *The operator D extends to a continuous invertible operator from $W_1^{q,0}(U)$ onto $B^q(U)$ with $DM = RD$.*

(We remark that by distribution theory [15], p. 29, D^{-1} can be given in terms of the Cauchy transform $(D^{-1}g)(w) = -\pi^{-1}\int_U g(z)(z-w)^{-1}dA(z)$.) Putting this all together, we are now able to convert our Bergman space problem to a Sobolev space problem as follows: Let $\mathcal{M}$ be S-invariant with

$$A^p(G) \subset \mathcal{M} \subset A^p(G\backslash K).$$

Taking annihilators, we obtain

$$B^q(G\backslash K) \subset \mathcal{M}^\perp \subset B^q(G)$$

and $R(\mathcal{M}^\perp) \subset \mathcal{M}^\perp$. Applying D^{-1} and Proposition 2.2 we have

$$W_1^{q,0}(G\backslash K) \subset D^{-1}(\mathcal{M}^\perp) \subset W_1^{q,0}(G)$$

and the subspace $D^{-1}(\mathcal{M}^\perp)$ is M-invariant. The rest of the paper will be dedicated to the description of $D^{-1}\mathcal{M}^\perp$ for which we will need a detailed understanding of the zero sets of functions in $W_1^{q,0}(G)$. Our analysis separates into two distinct cases; $1 < p < 2$, which will be taken up in Section 3, and the harder case $p \geq 2$, which will be examined in Section 4. Once $D^{-1}\mathcal{M}^\perp$ is known, we transfer back to $\mathcal{M}$ by the above recipe.

3. The Case $1 < p < 2$

Since $1 < p < 2$, then $q > 2$ and $W_1^{q,0}(G)$ is a Banach algebra of continuous functions. We will now utilize that fact that K lies on a simple compact C^1 arc to prove:

Theorem 3.1. *$D^{-1}\mathcal{M}^\perp$ is an ideal of $W_1^{q,0}(G)$.*

To prove Theorem 3.1, we need the following well known approximation lemma which uses very strongly the fact that the arc containing K is both simple and C^1 and is false if the arc is only piecewise C^1. We let γ be the simple, compact C^1 arc which contains K, by which we mean γ has a continuously differentiable parameterization $\alpha : [\alpha, \beta] \to \gamma$ such that $\alpha'(t)$ is non-vanishing on $[\alpha, \beta]$.

Lemma 3.2. *Let $\psi \in C^\infty(\mathbb{C})$. Given $\varepsilon > 0$, there is a polynomial $p(z)$ and $\Phi \in C^1(\mathbb{C})$ with $\Phi = \psi$ on γ and with $|p - \Phi| < \varepsilon$ and $|\nabla p - \nabla\Phi| < \varepsilon$ on $\mathbb{C}$.*

Proof. Since γ is a simple C^1 arc, there is an interval $[a, b]$ in $\mathbb{R}$, two-dimensional neighborhoods U and V of γ and $[a, b]$, respectively, and a diffeomorphism F of V onto U with $F([\alpha, \beta]) = \gamma$. We assume without loss of generality that the component functions of F and F^{-1} have bounded derivatives on V and U respectively. Define $Y(t) = F(t, 0)$, $a \leq t \leq b$, and $\Gamma = F([a, b])$. Let $g : [a, b] \to \mathbb{C}$ be a C^1 extension of $\psi(Y(t))$, $\alpha \leq t \leq \beta$, which has compact support in (a, b).

Let $\varepsilon > 0$ be given. By Lavrentiev's theorem [11], there is a polynomial $q(z)$ with

$$\left| q(z) - \frac{g'(Y^{-1}(z))}{Y'(Y^{-1}(z))} \right| < \varepsilon$$

for all $z \in \Gamma$. Let $p(z)$ be a polynomial with $p' = q$ and $p(Y(a)) = 0$. Then $\frac{d}{dt}p(Y(t)) = q(Y(t))Y'(t)$ so

$$\left| \frac{d}{dt}p(Y(t)) - g'(t) \right| = |q(Y(t))Y'(t) - g'(t)| = \left| Y'(t) \right| \left| q(Y(t)) - \frac{g'(t)}{Y'(t)} \right| \leq M\varepsilon$$

for all $a \le t \le b$, where $M = \sup\{|Y'(t)| : a \le t \le b\}$. Thus, for $a \le t \le b$,

$$|p(Y(t)) - g(t)| = \left| \int_a^t \frac{d}{dt} p(Y(s)) - g'(s) \right| \le (t-a)M\varepsilon \le (b-a)M\varepsilon.$$

Let $F^{-1} = (h_1, h_2)$ and define φ on U by

$$\varphi(x,y) = p(x+iy) - p(Y(h_1(x,y))) + g(h_1(x,y)).$$

Then, $\varphi = \psi$ on γ, $|p(x+iy) - \varphi(x,y)| \le (b-a)M\varepsilon$ for all $(x,y) \in U$, and

$$\left| \frac{\partial \varphi}{\partial x} - \frac{\partial p}{\partial x} \right| = \left| g'(h_1(x,y)) \frac{\partial h_1}{\partial x} - p'(Y(h_1(x,y)))Y'(h_1(x,y)) \frac{\partial h_1}{\partial x} \right| =$$

$$\left| \frac{\partial h_1}{\partial x} \right| \left| g'(h_1(x,y)) - p'(Y(h_1(x,y)))Y'(h_1(x,y)) \right| \le AM\varepsilon$$

on U, where $A = \sup\{|\frac{\partial h_1}{\partial x}| : z \in U\}$. Similarly

$$\left| \frac{\partial \varphi}{\partial y} - \frac{\partial p}{\partial y} \right| \le BM\varepsilon$$

on U, where $B = \sup\{|\frac{\partial h_1}{\partial y}| : z \in U\}$.

Let $\eta \in C_0^\infty(U)$ with $\eta = 1$ in a neighborhood of γ. Define Φ on $\mathbb{C}$ by

$$\Phi = \varphi\eta + (1-\varphi)p.$$

Then $\Phi = \psi$ on γ; and on $\mathbb{C}$, $|\Phi - p| = |\eta||p - \varphi| \le \|\eta\|_\infty (b-a)M\varepsilon$,

$$\left| \frac{\partial \Phi}{\partial x} - \frac{\partial p}{\partial x} \right| = \left| \frac{\partial \eta}{\partial x} \right| |p - \varphi| + |\eta| \left| \frac{\partial \varphi}{\partial x} - \frac{\partial p}{\partial x} \right| \le \left\| \frac{\partial \eta}{\partial x} \right\|_\infty (b-a)M\varepsilon + \|\eta\|_\infty AM\varepsilon,$$

and

$$\left| \frac{\partial \Phi}{\partial y} - \frac{\partial p}{\partial y} \right| \le \left\| \frac{\partial \eta}{\partial y} \right\|_\infty (b-a)M\varepsilon + \|\eta\|_\infty AM\varepsilon. \quad \square$$

Remark: In these next two proofs, we will use the fact that for $q > 2$, a function $f \in W_1^q$ belongs to $W_1^{q,0}(U)$ if any only if $f = 0$ on the complement of U [6], p. 313 - 314.

Proof of Theorem 3.1

To show $D^{-1}\mathcal{M}^\perp$ is an ideal of $W_1^{q,0}(G)$, it suffices to show $\psi D^{-1}\mathcal{M}^\perp \subset D^{-1}\mathcal{M}^\perp$ for all $\psi \in C_0^\infty(G)$. To this end, let $\psi \in C_0^\infty(G)$, $\varepsilon > 0$ be given, and p and Φ be as in Lemma 3.2.

If $f \in D^{-1}\mathcal{M}^\perp$, then $\Phi f \in W_1^{q,0}(G)$ with $\psi f - \Phi f = 0$ on K. So by the above remark, $\psi f - \Phi f \in W_1^{q,0}(G \backslash K) \subset D^{-1}\mathcal{M}^\perp$, thus $\mathrm{dist}(\psi f, D^{-1}\mathcal{M}^\perp) = \mathrm{dist}(\Phi f, D^{-1}\mathcal{M}^\perp)$. Since $pf \in D^{-1}\mathcal{M}^\perp$, then

$$\mathrm{dist}(\Phi f, D^{-1}\mathcal{M}^\perp) \le \|pf - \Phi f\|_q \le C\varepsilon \|f\|_q,$$

hence $\psi f \in D^{-1}\mathcal{M}^\perp$ Λ.

Proof of Theorem 1.1:

By our discussion above, $D^{-1}\mathcal{M}^\perp$ is an ideal of $W_1^{q,0}(G)$ that contains $W_1^{q,0}(G \backslash K)$. Let

$$Z_\mathcal{M} = \{z : f(z) = 0 \ \ \forall f \in D^{-1}\mathcal{M}^\perp\}$$

and note that $Z_\mathcal{M}$ is closed, and since $W_1^{q,0}(G \backslash K) \subset D^{-1}\mathcal{M}^\perp$, then $Z_\mathcal{M} \subset K$. By the above remark, $D^{-1}\mathcal{M}^\perp \subset W_1^{q,0}(G \backslash Z_\mathcal{M})$. For the other inclusion we let $\varphi \in C_0^\infty(G \backslash Z_\mathcal{M})$. Using Banach algebra techniques of Sarason [12], p. 41, Lemma 2, one finds a $g \in D^{-1}\mathcal{M}^\perp$

with $g = 1$ on $\text{supp}(\varphi)$ and thus $\varphi = g\varphi \in D^{-1}\mathcal{M}^{\perp}$. Hence $D^{-1}\mathcal{M}^{\perp} = W_1^{q,0}(G\backslash Z_{\mathcal{M}})$ and $\mathcal{M} = A^p(G\backslash Z_{\mathcal{M}})$ Λ.

4. The Case $p \geq 2$

For $p \geq 2$, $W_1^{q,0}(G)$ is not an algebra of continuous functions and thus the situation becomes more complicated. We will still describe $D^{-1}\mathcal{M}^{\perp}$ in terms of its zero set on K except that this set will not always be closed. To understand the zero sets of Sobolev functions, we use capacity.

4.1. **Capacity.** Following [2], we define the *q-capacity* C_q of a compact set F by

$$C_q(F) = \inf \|u\|_q,$$

where the infimum is taken over the all real-valued functions $u \in C_0^{\infty}$ with $u \equiv 1$ on F. We extend this definition to arbitrary sets E by

$$C_q(E) = \sup\{C_q(F) : F \subset E, \ \ F \text{ compact}\}$$

and define the *exterior capacity* $C_q^*(E)$ of an arbitrary set E by

$$C_q^*(E) = \inf\{C_q(G) : G \supset E, \ \ G \text{ open}\}.$$

A set E is said to *capacitable* if $C_q(E) = C_q^*(E)$. One notes [2] that C_q^* is a monotone, subadditive set function and that the Borel sets are capacitable. Recalling the definition of quasi-closed, one argues (using the fact that Borel sets are capacitable) that a quasi-closed set is capacitable, as is the difference of any two quasi-closed sets. We also say a property holds *quasi-everywhere* if the set for which it fails has exterior capacity zero.

Since functions in W_1^q, for $q \leq 2$, are not always continuous, we introduce a suitable substitution. A complex-valued function f is *quasi-continuous* if for every $\varepsilon > 0$ there is an open set W with $C_q(W) < \varepsilon$ and $f|_{\mathbb{C}\backslash W}$ continuous. One can show [2], Lemma 1, that every $f \in W_1^q$ has a quasi-continuous representative and this next result of Bagby [2], Theorem 4, describes $W_1^{q,0}(U)$ in terms of zero sets.

Proposition 4.1. *Let $u \in W_1^q$ be quasi-continuous. Then $u \in W_1^{q,0}(U)$ if and only if u vanishes quasi-everywhere off of U.*

4.2. **Invariant Subspaces.** We are now in a position to discuss invariant subspaces for $p \geq 2$. For a quasi-closed set $E \subset \mathbb{C}$, we find an increasing sequence of compact sets

$$F_1 \subset F_2 \subset \cdots \subset E$$

with $C_q(F_n) \to C_q(E)$. Since $A^p(G\backslash F_n)$ increases with n, we can define the S-invariant subspace

$$\mathcal{M}(E) = \overline{\bigcup_{n\geq 1} A^p(G\backslash F_n)}^{L^p}. \tag{4.1}$$

Proposition 4.2.
(i) $\mathcal{M}(E)$ is independent of the choice of $\{F_n\}$.
(ii) If E_1, E_2 are quasi-closed, then $\mathcal{M}(E_1) \subset \mathcal{M}(E_2) \Leftrightarrow C_q(G \cap (E_1\backslash E_2)) = 0$. Thus $\mathcal{M}(E_1) = \mathcal{M}(E_2) \Leftrightarrow C_q(G \cap (E_1 \Delta E_2)) = 0$.

Proof. (i) For a quasi-closed set $E \subset G$, let

$$W_q(E) \equiv D^{-1}(\mathcal{M}(E)^{\perp}) = \bigcap_{n\geq 1} W_1^{q,0}(G\backslash F_n)$$

and notice, by Proposition 4.1, that a quasi-continuous $f \in W_1^{q,0}(G)$ belongs to $W_q(E)$ if and only if $f = 0$ quasi-everywhere on E. Thus $W_q(E)$, and hence $\mathcal{M}(E)$, is independent of the choice of $\{F_n\}$.

(ii) If $C_q(G \cap (E_1\backslash E_2)) = 0$, then, by the above comments, $W_q(E_2) \subset W_q(E_1)$, hence $\mathcal{M}(E_1) \subset \mathcal{M}(E_2)$. If $C_q(G \cap (E_1\backslash E_2)) > 0$, one argues, using Proposition 4.1, the above remarks, and a result of Bagby [2], Theorem 3, that $W_q(E_2)\backslash W_q(E_1) \neq \emptyset$, and hence $\mathcal{M}(E_1)\backslash\mathcal{M}(E_2) \neq \emptyset$. □

As mentioned in the introduction, the union in (4.1) does not always collapse down to a single $A^p(G\backslash F)$.

Proposition 4.3. *For $p \geq 2$, not every $\mathcal{M}$ is of the form $A^p(G\backslash F)$.*

Proof. Fix $1 < q \leq 2$ and let G be a disk of radius 2 centered about the origin and $K = [0,1]$. Let $B \subset [0,1]$ be constructed in the same manner as the Cantor set except that the intervals removed (a_n, b_n) are such that $\sum_{n\geq 1} C_q(a_n, b_n) < C_q[0,1]$. (This is justified since $C_q(a,b)^q \simeq (b-a)^{2-q}$ if $q < 2$ and $C_2(a,b)^2 \simeq (log(2/(b-a))^{-1}$ [13], and [7], p. 115, Proposition 6.) Set $E = [0,1]\backslash B = \cup_{n\geq 1}(a_n, b_n)$ and notice that E is open and dense in $[0,1]$ with $C_q(E) < C_q[0,1]$. A straightforward argument shows that E is quasi-closed and $C_q(E\Delta F) > 0$ for any closed set F. Setting $\mathcal{M} = \mathcal{M}(E)$ and using Proposition 4.2, we are done. □

Using the proof of Theorem 3.1, one can prove $\psi D^{-1}\mathcal{M}^{\perp} \subset D^{-1}\mathcal{M}^{\perp}$ for every $\psi \in C^{\infty}$. We will use this to ultimately show $D^{-1}\mathcal{M}^{\perp} = W_q(E)$ and hence $\mathcal{M} = \mathcal{M}(E)$ for some quasi-closed $E \subset K$. To accomplish this, we let $f \in W_1^{q,0}(G)$ (assumed to be quasi-continuous), and define

$$[f] = \text{span}\{\varphi f : \varphi \in C^{\infty}\}.$$

If we define $Z_f = f^{-1}(0)$, we see (using the fact that $f^{-1}(F)$ is quasi-closed for closed F and quasi-continuous f) that Z_f is quasi-closed and, by the proof of Proposition 4.2, $[f] \subset W_q(Z_f)$. To keep our exposition clear, we defer the proofs of the following two results to the very end.

Lemma 4.4. *If $g, h \in W_1^{q,0}(G)$ with $|g(z)| \leq |h(z)|$ a.e., then $g \in [h]$.*

Lemma 4.5. *If $f \in W_1^{q,0}(G)$ is quasi-continuous, then $[f] = W_q(Z_f)$.*

Assuming these two facts, one can now show that $D^{-1}\mathcal{M}^{\perp} = W_q(E)$, for some quasi-closed $E \subset K$.

Corollary 4.6. *There exists a quasi-continuous $f \in W_1^{q,0}(G)$ with*

$$D^{-1}\mathcal{M}^{\perp} = [f] = W_q(Z_f).$$

Proof. Since $D^{-1}\mathcal{M}^{\perp}$ is separable, there is a sequence of quasi-continuous functions $\{f_n : n \geq 1\}$ in $W_1^{q,0}(G)$ with

$$\mathcal{D}^{-1}\mathcal{M}^{\perp} = \text{span}\{[f_n] : n \geq 1\}.$$

By [4], p. 316, $|f_n| \in W_1^{q,0}(G)$, and by Lemma 4.4, $[|f_n|] = [f_n]$. Thus we may assume $f_n \geq 0$. For each $n \geq 1$, let $\varepsilon_n = \|f_n\|_q^{-1} 2^{-n}$ and define $f = \sum_n \varepsilon_n f_n \in W_1^{q,0}(G)$. Assuming

f is quasi-continuous, we see that $Z_f = Z_{\cap Z_{f_n}}$ quasi-everywhere. (This will follow from the fact that if p_n is the n-th partial sum, then $C_q(f - p_n \geq \varepsilon) \leq \varepsilon^{-q}\|f - p_n\|_q$ [2], Theorem 2(i) and hence a subsequence of p_n will converge to f quasi-everywhere.) Thus

$$f \in \operatorname{span}\{[f_n] : n \geq 1\} = \mathcal{D}^{-1}\mathcal{M}^{\perp} \subset W_q(Z_f),$$

and hence, by Lemma 4.5, $[f] = \mathcal{D}^{-1}\mathcal{M}^{\perp} = W_q(Z_f)$. □

Proof of Theorem 1.2.

By Corollary 4.6 there is quasi-closed set E with $D^{-1}(\mathcal{M}^{\perp}) = W_q(E)$. Since $\mathcal{M}(E) \subset A^p(G \backslash K)$, we can apply Proposition 4.2 to assume $E \subset K$. Λ

4.3. **Weak Convergence and Cut-off Functions.** In this last part, we prove Lemma 4.4 and Lemma 4.5. To do this, we will use cut-off functions and and weak topology techniques of [9].

By [1], Theorem 3.10, p. 50, the dual space of $W_1^{q,0}(G)$, denoted by $W_{-1}^p(G)$, is the set of linear functionals of the form

$$\ell(u) = \int_G \left(v_0 u + v_1 \frac{\partial u}{\partial x} + v_2 \frac{\partial u}{\partial y} \right) dA, \quad v_0, v_1, v_2 \in L^p(G), \tag{4.2}$$

and the norm of ℓ satisfies

$$\|\ell\| \leq C \left(\sum_{k=0}^{2} \|v_k\|^p_{L^p(G)} \right)^{1/p}. \tag{4.3}$$

Proposition 4.7. *Let $\{f_n : n \geq 1\}$ be a sequence of functions in $W_1^{q,0}(G)$. If $f_n \to 0$ a.e. and $\|f_n\|_q$ is uniformly bounded in n, then $f_n \to 0$ weakly in $W_1^{q,0}(G)$.*

Proof. Since $\{f_n : n \geq 1\}$ is uniformly bounded in $L^q(G)$-norm, we can apply Egorov's theorem to obtain $\int f_n \varphi dA \to 0$ for all $\varphi \in C_0^\infty(G)$. Since $\{f_n : n \geq 1\}$ is uniformly bounded in Sobolev norm, we can apply the Banach-Alaoglu theorem (since $W_1^{q,0}(G)$ is reflexive) to get that every subsequence has a weakly convergent subsequence that converges to h (h will depend on the subsequence). Thus if $\ell_\varphi \in W_{-1}^p(G)$ is defined by $\ell_\varphi(u) = \int u\varphi dA$, where $\varphi \in C_0^\infty(G)$, then

$$\ell_\varphi(h) = \lim_{j\to\infty} \ell_\varphi(f_{n_{k_j}}) = 0.$$

Thus $\int h\varphi dA = 0$ for all $\varphi \in C_0^\infty(G)$, hence $h = 0$. Thus $f_n \to 0$ weakly. □

The proof of Lemma 4.4 will depend on this next lemma for which we mention a few technicalities which can be found in [15], p. 55. (We thank A. Aleman for showing us this proof.) Given a function $u \in L^p$, $1 \leq p \leq \infty$, and $r > 0$ we define

$$u_r(w) = \frac{1}{2\pi i r^2} \int_{|z|=r} u(z + w)dz$$

and notice from Fubini's theorem that the line integral exists for almost all w and that $u_r \in L^p$ when $u \in L^p$, $1 \leq p \leq \infty$. One also shows [15], p. 54 - 55, (using Green's theorem) that the Cauchy transform

$$(Cu_r)(w) = -\frac{1}{\pi} \int (z - w)^{-1} u_r(z)dA(z)$$

of u_r is given by

$$(Cu_r)(w) = \frac{1}{\pi r^2} \int_{|y-w|<r} u(y)dA(y).$$

In particular, by the Hardy-Littlewood inequality [14], p. 5, $\|Cu_r\|_{L_p} \leq \|u\|_{L^p}$ and

$$\lim_{r\to 0}(Cu_r)(w) = u(w) \text{ a.e.} \tag{4.4}$$

Finally, we mention that if $u \in W_1^q$ (with compact support), then

$$u_r(w) = \frac{1}{\pi r^2}\int_{|y-w|<r} \overline{\partial} u(y)dA(y). \tag{4.5}$$

We will also need the following approximation lemma: For $u \in L^\infty$, we let u_ε, $\varepsilon > 0$, be a mollification of u [1], p. 52. Note that $\|u_\varepsilon\|_\infty \leq \|u\|_\infty$ for all $\varepsilon > 0$ and that $u_\varepsilon \to u$ pointwise a.e. as $\varepsilon \to 0$. Also notice that if $\overline{\partial} u \in L^\infty$, then an easy calculation shows that $\|\overline{\partial} u_\varepsilon\|_\infty \leq \|\overline{\partial} u\|_\infty$.

Lemma 4.8. *If $u \in L^\infty$ and $f \in W_1^{q,0}(G)$ with $uf \in W_1^{q,0}(G)$. Then $uf \in [f]$.*

Proof. First notice that $Cu_r \in L^\infty$ and that $\overline{\partial} Cu_r = u_r \in L^\infty$. Thus, for fixed $r > 0$, if h_ε is a mollification of Cu_r, [1], p. 29, then $h_\varepsilon f \in [f]$ and $h_\varepsilon f \to Cu_r f$ a.e. One also notices (from the discussion above) that $h_\varepsilon f$ is uniformly bounded in Sobolev norm and thus by Proposition 4.7 $Cu_r f \in [f]$.

Since $Cu_r f \to uf$ pointwise a.e. (4.4), we see by Proposition 4.7, that it suffices to show $\|Cu_r f\|_q \sim \|\overline{\partial}(Cu_r f)\|_{L^q}$ (see (2.1)) remains uniformly bounded in r. Notice that

$$\overline{\partial}(Cu_r f) = Cu_r \overline{\partial} f + u_r f \tag{4.6}$$

and using the Hardy-Littlewood inequality again,

$$\|Cu_r \overline{\partial} f\|_{L^q} \leq \|Cu_r\|_\infty \|f\|_{L^q} \leq \|u\|_\infty \|f\|_{L^q}.$$

To bound (in L^q norm) the second term of (4.6), we write

$$u_r f(w) = (uf)_r(w) + \frac{1}{2\pi i r^2}\int_{|z|=r} u(z+w)(f(w) - f(z+w))dz.$$

Since $uf \in W_1^q$ we have from (4.5) that

$$(uf)_r(w) = \frac{1}{\pi r^2}\int_{|w-z|<r} \overline{\partial}(uf)(y)dA(y)$$

which, again by the Hardy-Littlewood inequality, has L^q norm bounded by a constant multiple of $\|uf\|_q$. Finally, we estimate

$$\left| \frac{1}{2\pi i r^2}\int_{|z|=r} u(z+w)(f(w) - f(z+w))dz \right| \leq \|u\|_\infty \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(re^{it}+w) - f(w)|}{r} dt.$$

Using [14], Theorem 3, p. 135 and Proposition 3, p. 139, we see that that the L^q modulus of continuity of a function in W_1^q is $O(|r|)$. From this we deduce that

$$\left\| \frac{|f(re^{it}+w) - f(w)|}{r} \right\|_{L^q}$$

is uniformly bounded in r. An application of Fubini's theorem completes the proof. □

Proof of Lemma 4.4

Let $f = h$ and $u = gh^{-1}$ if $h \neq 0$ and zero otherwise. Note that $u \in L^\infty$ with $uf \in W_1^{q,0}(U)$. Now apply Lemma 4.8 to get $uf = g \in [h]$. Λ

Remark: The following fact will be important for what follows [4], p. 316: If $f \in W_1^q$ is real-valued, then $f^+ = \max(f, 0) \in W_1^q$ with

$$\int |\nabla f^+|^q dA = \int_{f>0} |\nabla f|^q dA \le \int |\nabla f|^q dA.$$

Lemma 4.9. *$W_q(Z_f) \cap L^\infty$ is dense in $W_q(Z_f)$.*

Proof. Let $g \in W_q(Z_f)$. It follows from Lemma 4.4 that $[|g|] = [g] \subset W_q(Z_f)$, thus we may assume $g \ge 0$. For an integer $M > 0$, let $g_M = \min\{g, M\}$ and apply [4], p. 316, and the above remark to get $g_M \in W_q(Z_f) \cap L^\infty$ with $\|g_M\|_q \le \|g\|_q$. Since $g_M \to g$ a.e. as $M \to \infty$, we can apply Proposition 4.7 to show $g_M \to g$ weakly. □

Proof of Lemma 4.5:

This proof is a modification of in idea found in [7], p. 89 - 90 and very similar to [9], Lemma 4.2. Clearly $[f] \subset W_q(Z_f)$. If f_1 denotes the cut-off function $f_1 = \min\{|f|, 1\}$ then $Z_f = Z_{f_1}$ and since $f_1 \le f$ it follows from Lemma 4.4 that

$$[f_1] \subset [f] \subset W_q(Z_f) = W_q(Z_{f_1}).$$

Thus we may assume $0 \le f \le 1$. Moreover it follows from Lemma 4.9 that it suffices to show $W_q(Z_f) \cap L^\infty \subset [f]$.

To this end, let $g \in W_q(Z_f) \cap L^\infty$ be quasi-continuous. By Lemma 4.4 we may assume $g \ge 0$. For each positive integer n let

$$g_n = \max\{g - \frac{1}{n}, 0\}$$

and notice that $g_n \to g$ a.e., g_n is uniformly bounded in Sobolev norm (see the above remark), and $g_n \in W_1^{q,0}(G)$ (Proposition 4.1). So applying Proposition 4.7, it suffices to show $g_n \in [f]$.

For what follows, we fix a positive integer n. For $t \ge 0$ we define

$$N_t = \{z : g_n(z) \ne 0, f(z) \le t\}.$$

The functions f and g are quasi-continuous, hence the sets

$$M_t = \{z : g(z) \ge \frac{1}{n}, f(z) \le t\}$$

are quasi-closed and for each $t \ge 0$ they satisfy $N_t \subset M_t$. Now $M_0 \subset Z_f \backslash Z_g$, hence by assumptions on g and the fact that the M_t are decreasing (as $t \to 0$) and quasi-closed, we can apply a result of Fuglede, [5], Lemma 2 (really just a generalization of the fact that for compact sets $K_i \downarrow K$ implies $C_q(K_i) \to C_q(K)$, to quasi-closed sets), to obtain

$$C_q^*(N_t) \le C_q^*(M_t) \to C_q^*(M_0) = 0$$

as $t \to 0$. By [2], Theorem 2(i), we can find a family $0 \le w_t \le 1$ of functions in W_1^q with $w_t = 1$ quasi-everywhere on N_t and $\|w_t\|_q \to 0$ as $t \to 0$. For $\delta > 0$, notice that

$$\|\nabla(f+\delta)^{-1}\|_{L^q(G)} \le \|(f+\delta)^{-2}\|_\infty \|\nabla f\|_{L^q} \le \delta^{-2}\|\nabla f\|_{L^q}.$$

For $t, \delta > 0$, define $u_{t,\delta}$ by

$$u_{t,\delta} = \frac{(1 - w_t)g_n}{f + \delta}. \tag{4.7}$$

Since $W_1^q \cap L^\infty$ is an algebra [7], p. 48, we have $u_{t,\delta} \in W_1^q \cap L^\infty$. Applying Lemma 4.8, we have $f u_{t,\delta} \in [f]$.

We now show that we can choose a sequence $t_j \to 0$ with $fu_{t_j,\delta(t_j)}$ uniformly bounded in Sobolev norm and converging to g_n a.e. Once this has been established, we apply Proposition 4.7 to show $g_n \in [f]$ and the proof will be finished.

To control the Sobolev norm of $g_n - fu_{t,\delta}$ in t and δ we let $\varphi_t(x)$ be a smooth increasing function on $[0,\infty)$ with $\varphi_t(0) = t/4$ and $\varphi_t(x) = x$ for all $x > t/2$. Since $w_t = 1$ quasi-everywhere on N_t, then

$$g_n - fu_{t,\delta} = w_t g_n + \frac{\delta(1-w_t)g_n}{f+\delta} = w_t g_n + \frac{\delta(1-w_t)g_n}{\varphi_t(f)+\delta}. \tag{4.8}$$

Let $\psi_{t,\delta}(x) = (\varphi_t(x)+\delta)^{-1}$ and note that since $\varphi_t + \delta \geq t/4$, then $\|\psi_{t,\delta}\|_\infty \leq 4t^{-1}$ and $\|\psi'_{t,\delta}\|_\infty \leq 16\|\varphi'_t\|_\infty t^{-2}$. First note that

$$\|\nabla(\varphi_t \circ f + \delta)^{-1}\|_{L^q(G)} = \|\nabla(\psi_{t,\delta} \circ f)\|_{L^q(G)} \tag{4.9}$$

$$\leq \|\psi'_{t,\delta}\|_\infty \|\nabla f\|_{L^q} \leq 16\|\varphi'_t\|_\infty \|\nabla f\|_{L^q} t^{-2} \leq C_t t^{-2}, \tag{4.10}$$

where C_t is a constant independent of δ.

Looking back at (4.8), we see that

$$\|g_n - fu_{t,\delta}\|_q \leq \|w_t g_n\|_q + \|\delta g_n(1-w_t)\psi_{t,\delta}(f)\|_q. \tag{4.11}$$

The first term on the right hand side of (4.11) is uniformly bounded in t since

$$\|w_t g_n\|_q \leq \|w_t\|_\infty \|g_n\|_q + \|g_n\|_\infty \|w_t\|_q, \tag{4.12}$$

$0 \leq w_t \leq 1$ for all $t > 0$, and $\|w_t\|_q \to 0$. The second term on the right hand side of (4.11) is bounded by

$$\delta\|(1-w_t)g_n\|_\infty \|\nabla\psi_{t,\delta}(f)\|_{L^q} + \delta\|(1-w_t)g_n\|_q \|\psi_{t,\delta}(f)\|_\infty. \tag{4.13}$$

By (4.12), the quantities $\|(1-w_t)g_n\|_q$ and $\|(1-w_t)g_n\|_\infty$ are uniformly bounded in t and applying (4.9) and the fact that $\|\psi_{t,\delta}\|_\infty \leq 4t^{-1}$ to (4.13), we see that

$$\|\delta g_n(1-w_t)\psi_{t,\delta}(f)\|_q \leq \delta D_t t^{-2}, \tag{4.14}$$

where D_t is a constant independent of δ. Letting $\delta(t) = t^2(D_t+1)^{-1}$ we see that $\delta(t) \to 0$ as $t \to 0$ and that (4.14) and hence $g_n - fu_{t,\delta(t)}$ is uniformly bounded in Sobolev norm for $t \to 0$.

To conclude, we show that $u_{t_j,\delta(t_j)}f \to g_n$ a.e. for some sequence $t_j \to 0$. Since $\|w_t\|_q \to 0$ as $t \to 0$, there is a sequence $t_j \to 0$ with $w_{t_j} \to 0$ a.e. as $j \to \infty$. Thus

$$u_{t_j,\delta(t_j)}f = (1-w_{t_j})g_n \frac{f}{f+\delta(t_j)} \to g_n$$

a.e. on the complement of Z_f. But this all we need since $u_{t_j,\delta(t_j)}f = g_n = 0$ a.e. on Z_{g_n} and $Z_f \backslash Z_{g_n}$ has measure zero. Λ

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