2015

One-Step Bond Pricing

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This paper re-works the traditional formula for pricing a bond with a present value annuity and a single discounted cash flow into a "one-step" bond pricing formula. The new derivation allows for a clear presentation of the relationship between a bond's coupon yield and its yield to maturity and a quick means for pricing a bond.

INTRODUCTION

In relation to the teaching of time value of money, bond pricing offers a context for using a present value annuity to value a bond's coupons in conjunction with a single discounted cash flow to value the bond's par payment at maturity. From a pedagogy perspective, not much better material exists for reinforcing time value of money concepts.

However, being able to determine that a bond should sell at a discount (premium) when the coupon yield is below (above) the yield to maturity is not readily apparent. Similarly, understanding that a bond should sell at par when the coupon yield and yield to maturity are equal is also not very apparent.

Logically, it can be stated that the yield to maturity is the return that the market desires from the bond based on its risk and that the coupon yield can either under-, exactly- or over-compensate for this risk which leads to the bond price being at a discount, par, or a premium respectively. Such a statement is correct, but a mathematical identity that demonstrates this relationship can significantly reinforce such a statement.

In this paper, a "one-step" bond pricing formula is presented in which the par value of the bond (usually $1,000.00) is multiplied by a factor to produce the bond price. Within the structure of this bond pricing factor, the relationship between the coupon yield and the yield to maturity become very transparent. The "one-step" model is demonstrated in the next section with the paper concluding in the following section.

ONE STEP BOND PRICING

The traditional approach to bond pricing can be found in any number of textbooks (e.g. Ross, Westerfield, and Jordan, 2013) with the bond price being a combination of discounted coupons (c) and a discounted par payment (Par) with a yield to maturity of "k" and a maturity of "N" periods:

$$Bond\ price = \frac{c}{k}\left[1 - \frac{1}{(1+k)^N}\right] + \frac{Par}{(1+k)^N} \tag{1}$$

In this form, the effect of the coupon yield is hidden, but can be introduced with the identity ($c = k_C \times Par$, where "k_C" is the coupon yield):

$$Bond\ price = \frac{Par \times k_C}{k}\left[1 - \frac{1}{(1+k)^N}\right] + \frac{Par}{(1+k)^N} \tag{2}$$

If one continues to manipulate equation (2), an equation emerges that has the par value of the bond multiplied by a single factor that has the structure of a present value annuity:

$$Bond\ price = Par \times \left\{1 + \frac{(k_C - k)}{k}\left[1 - \frac{1}{(1+k)^N}\right]\right\} \tag{3}$$

Equation (3) is the "one-step" bond pricing technique. The factor in the brackets (call it the "bond price multiplier") is simply one plus a present value annuity of the difference between the coupon yield and the yield to maturity. In this form, it is very easy to see the relationship between the coupon yield and the yield to maturity just by calculating ($k_C - k$):

($k_C - k$) > 0 means the bond price multiplier > 1 and the bond sells at a premium
($k_C - k$) = 0 means the bond price multiplier = 1 and the bond sells at par
($k_C - k$) < 0 means the bond price multiplier < 1 and the bond sells at a discount

The one-step method is similar to the "differential approach" of Hahn and Lange (2008). Hahn and Lange produce an equation (adapted to the notation in this presentation) for the bond price using sigma notation:

$$Bond\ price = Par + \sum_{t=1}^{N} \frac{(k_C - k) \times Par}{(1+k)^t} \tag{4}$$

Introduce an annuity equation for the summation portion of the formula and then factor the bond's par value (Par) to reproduce equation (3):

$$Bond\ price = Par + \frac{(k_C - k) \times Par}{k}\left[1 - \frac{1}{(1+k)^N}\right]$$

$$Bond\ price = Par \times \left\{1 + \frac{(k_C - k)}{k}\left[1 - \frac{1}{(1+k)^N}\right]\right\} \tag{5}$$

The one-step method is derived very quickly with a calculator or with an annuity table. Let $PV(N, k)$ stand for the present value factor for an annuity assuming a discount rate of "k" for "N" periods.

$$PV(N,k) = \frac{1}{k}\left[1 - \frac{1}{(1+k)^N}\right] \tag{6}$$

The one-step method becomes:

$$Bond\ price = Par \times \{1 + (k_C - k) \times PV(N,k)\} \tag{7}$$

Consequently, for classes in which calculating the bond price is not a priority, the one-step technique allows a quick means for calculating the bond price.

For example, current yield (= coupon ÷ bond price; coupon = $k_C \times Par$) is an approximation for yield to maturity. An annuity table for N = 10 and three different interest rates: $PV(10, 7\%) = 7.0236$, $PV(10, 8\%) = 6.7101$, $PV(10, 9\%) = 6.4177$, with a coupon yield set at the middle interest rate value (8% in this example) provides all the information necessary to demonstrate how good of an approximation current yield is relative to "k". Set the bond (assume a par value of \$1,000.00) to sell at a discount by having the coupon yield less than the yield to maturity ($k_C = 8\%$ and $k = 9\%$):

$$Bond\ price = \$935.82 = \$1{,}000.00 \times \{1 + (8\% - 9\%) \times 6.4177\}$$

$$Current\ yield = 8.55\% = \frac{\$80.00}{\$935.82} \tag{8}$$

Notice, the current yield is not a very good approximation of the yield to maturity of 9% (i.e. the current yield is an under-estimation of the yield to maturity).

Set the bond to sell at par by having the coupon yield and the yield to maturity be equal ($k_C = 8\%$ and $k = 8\%$):

$$Current\ yield = 8.00\% = \frac{\$80.00}{\$1{,}000.00} \tag{9}$$

In this case, the current yield is an exact approximation of the yield to maturity.

Set the bond to sell at a premium by having the coupon yield greater than the yield to maturity ($k_C = 8\%$ and $k = 7\%$):

$$Bond\ price = \$1{,}070.24 = \$1{,}000.00 \times \{1 + (8\% - 7\%) \times 7.0236\}$$

$$Current\ yield = 7.47\% = \frac{\$80.00}{\$1{,}070.24} \tag{10}$$

In this case, the current yield is again not a very good approximation of the yield to maturity (i.e. the current yield is an over-estimation of the yield to maturity).

The lesson demonstrated is that the current yield becomes a better approximation of the yield to maturity when the bond sells at or close to par value. The direction of the approximation error is also dependent upon the bond selling at a premium (leads to an under-estimation) or at a discount (leads to an over-estimation). Although bond pricing was part of the lesson, the one-step method reduced the calculation portion of computing the bond price which allowed more emphasis on illustrating the ability of the current yield to approximate the yield to maturity.

The need to de-emphasize actual bond pricing while demonstrating bond price dynamics is very common when preparing students for licensure tests (e.g. the Series 7). The need also appears when lecturing an audience that has little to no financial background.

CONCLUSION

The one-step bond pricing method is a very quick means for finding a bond price relative to the traditional textbook method of using an annuity and a single discounted cash flow. The one-step method demonstrates readily the relationship of: ($k_C < k$) implying a discounted bond price, ($k_C = k$) implying a bond price equal to par, and ($k_C > k$) implying a premium bond price. The method is also useful in a course that may require bond pricing, but is not necessarily focused on bond pricing.

ENDNOTES

[1] The author wishes to thank Carol Lancaster for helpful feedback.

REFERENCES

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