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Abstract. In this paper, we examine the invariant subspaces (under the operator $f \to zf$) $\mathscr{M}$ of the Bergman space $L^p_a(G\backslash\mathbb{T})$ (where $1 < p < 2$, G is a bounded region in $\mathbb{C}$ containing $\bar{\mathbb{D}}$, $\mathbb{T}$ is the unit circle, and $\mathbb{D}$ is the unit disk) which contain the characteristic functions $\chi_{\mathbb{D}}$ and χ_G, i.e. the constant functions on the components of $G\backslash\mathbb{T}$. We will show that such $\mathscr{M}$ are in one-to-one correspondence with the invariant subspaces of the analytic Besov space AB_q (q is the conjugate index to p) and then use results of Shirokov to describe such $\mathscr{M}$. When $p \geq 2$ the situation becomes more complicated and capacity considerations are needed.

1. Introduction. For $1 < p < \infty$ and a bounded open set $U \subset \mathbb{C}$, the *Bergman space* $L^p_a(U)$ is the space of analytic functions f on U for which

$$\int_U |f(z)|^p \, dx \, dy < \infty.$$

The subspaces $\mathscr{M} \subset L^p_a(U)$ with $z\mathscr{M} \subset \mathscr{M}$ (We will call such subspaces *invariant subspaces.*) are so fantastically complicated that they defy a reasonable characterization. In this paper, we wish to continue an investigation begun in [2], [23] of the invariant subspaces

$$\chi_G \in \mathscr{M} \subset L^p_a(G\backslash K),$$

where K is a compact subset of a bounded region $G \subset \mathbb{C}$ and $\text{Area}(K) = 0$. When $1 < p < 2$ and $G\backslash K$ is connected, $\mathscr{M}$ has a relatively simple characterization as $\mathscr{M} = L^p_a(G\backslash E)$ for some closed $E \subset K$. When $p \geq 2$ and $G\backslash K$ is connected,

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not all $\mathscr{M}$ are of the form $L^p_a(G\backslash E)$ but instead take the form

$$\mathscr{M} = \overline{\bigcup_n L^p_a(G\backslash E_n)}^{L^p},$$

where $\{E_n\}$ is an increasing sequence of closed subsets of K.

When $G\backslash K$ is not connected, the problem becomes much more complicated. In this paper we begin to investigate this situation in the special case when G is a region containing the closure of the open unit disk $\mathbb{D}$, K is the unit circle $\mathbb{T}$, and the invariant subspace $\mathscr{M}$ has the property

$$\chi_G, \chi_{\mathbb{D}} \in \mathscr{M}.$$

(i.e. $\mathscr{M}$ contains the constants on the components of $G\backslash\mathbb{T}$).

Remark. Throughout this paper, a 'region' will be an open connected subset of the plane and a 'domain' will be a open subset of the plane (it need not be connected).

It will turn out, via annihilators and the Cauchy transform, that such invariant subspaces $\mathscr{M}$ will be in one-to-one correspondence with the invariant subspaces (under multiplication by ζ) of the analytic Besov space AB_q (q is the conjugate index to p) of Hardy space H^q functions with

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \left| \frac{f(\zeta) - f(\xi)}{\zeta - \xi} \right|^q |d\zeta||d\xi| < \infty.$$

When $1 < p < 2$, then $q > 2$ and AB_q becomes an Banach algebra of continuous functions on $\mathbb{T}$ and the invariant subspaces are the closed ideals of AB_q which have been characterized by Shirokov [26] as

$$\mathscr{F} = \mathscr{F}(E, I) \equiv \{f \in AB_q : f|_E = 0,\ f/I \in H^\infty\},$$

for some closed set $E \subset \mathbb{T}$ and inner function $I \in H^\infty(\mathbb{D})$. Moreover, if $I = BS_\mu$ is the usual factorization of the inner function I into a Blaschke product B, with zeros $\{a_k\}$, and a singular inner function S_μ, with positive singular measure μ, then we set

$$\operatorname{spec}(I) = \operatorname{clos}\{a_k\} \cup \operatorname{supp}(\mu).$$

With this notation, it is known (see below) that the ideal $\mathscr{F}(E, I) \neq (0)$ if and only if the following condition is satisfied:

$$\int_{\mathbb{T}} \log \operatorname{dist}(\zeta, E \cup \operatorname{spec}(I))|d\zeta| > -\infty. \tag{1.1}$$

Thus for $1 < p < 2$, every invariant subspace $\chi_{\mathbb{D}}, \chi_G \in \mathscr{M} \subset L^p_a(G\backslash\mathbb{T})$ can be written as $\mathscr{M}_{\mathscr{F}(E,I)}$ and we will show that $\mathscr{M}_{\mathscr{F}(E,I)} = L^p_a(G\backslash\mathbb{T})$ if and only if $\mathscr{F}(E,I) = 0$. Our first theorem identifies $\mathscr{M}_{\mathscr{F}(E,I)}$.

Theorem 1.1. *Let $1 < p < 2$ and $\chi_G, \chi_{\mathbb{D}} \in \mathscr{M} \subset L^p_a(G\backslash\mathbb{T})$ be invariant. Then there is a closed set $E \subset \mathbb{T}$ and an inner function $I \in H^\infty(\mathbb{D})$ with*

$$\mathscr{M} = \mathscr{M}_{\mathscr{F}(E,I)} = L^p_a(G\backslash E) \bigvee \left\{ \frac{\chi_{G\backslash\mathbb{D}}}{\phi} L^p_a(G) : \ \phi \text{ inner}, \ \frac{I}{\phi} \in H^\infty \right\}$$

Moreover, $\mathscr{M} \neq L^p_a(G\backslash\mathbb{T})$ if and only if condition (1.1) *is satisfied.*

Here we use the notation $A \bigvee B$ to denote the closed linear span of A and B. Notice that for an inner function ϕ and $|z| > 1$, we have $\phi(z) = \phi(z^*)^*$ $(a^* = 1/\bar{a})$ and hence $1/|\phi(z)| \le 1$ for $|z| > 1$. Thus $\chi_{G\backslash\mathbb{D}}/\phi \in L^p_a(G\backslash\mathbb{T})$. Throughout this paper when use the term inner function, we mean a bounded analytic function on the unit disk $\mathbb{D}$ which is unimodular a.e. on the unit circle $\mathbb{T}$ (in contrast to a inner function defined on a general domain). We can refine Theorem 1.1 as follows: If the inner function I is the least common multiple (see definition below) of the inner functions I_1 and I_2 (e.g. $I_1 = 1, I_2 = I$, or $I_1 = B$, $I_2 = S_\mu$), then

$$\mathscr{M}_{\mathscr{F}(E,I)} = L^p_a(G\backslash E) \bigvee \frac{\chi_{G\backslash\mathbb{D}}}{I_1} L^p_a(G) \bigvee \frac{\chi_{G\backslash\mathbb{D}}}{I_2} L^p_a(G).$$

For $p = 2$, the problem becomes more complicated since the analytic Besov space AB_2 (often called the Dirichlet space) is no longer an algebra of continuous (or even bounded) functions and the invariant subspaces of are not completely understood. It is a result of Beurling [6] that for $f \in AB_2$, the radial limit

$$\lim_{r\to 1} f(r\zeta)$$

exists quasi-everywhere (q.e.), that is to say everywhere except possibly on a set of Bessel capacity (see definition below) zero. For a set $E \subset \mathbb{T}$, define $AB_{2,E}$ to be the set of $f \in AB_2$ with radial limit zero q.e. on E. One shows [7] that $AB_{2,E}$ is a closed invariant subspace of AB_2 and it is an open question as to whether or not all invariant subspaces of AB_2 are of the form

$$\mathscr{F}_{E,I} \equiv IH^2 \cap AB_{2,E}$$

for some inner function I and set $E \subset \mathbb{T}$. We will show, as in the $q > 2$ case, that there is a one-to-one correspondence between the invariant subspaces of AB_2 and the invariant subspaces

$$\chi_{\mathbb{D}}, \chi_G \in \mathscr{M} \subset L^2_a(G\backslash\mathbb{T}).$$

We will then then identify $\mathscr{M}_{\mathscr{F}_{E,I}}$. Thus a complete description of the invariant subspaces of AB_2 as $\mathscr{F}_{E,I}$ will yield a complete description of the invariant subspaces $\mathscr{M}$. These results generalize to $p > 2$.

Before proceeding, we mention that the condition requiring both χ_G and $\chi_{\mathbb{D}}$ belong to $\mathscr{M}$ is not a superfluous one. Without it, the problems becomes nearly impossible to solve as can be seen by the following example: By [4], Corollary 6.9 and Proposition 5.4, given any $n \in \mathbb{N} \cup \{\infty\}$ there is an invariant subspace $\mathscr{N}_n$ of $L^2_a(\mathbb{D})$ with $\dim(\mathscr{N}_n/z\mathscr{N}_n) = n$. Consider the invariant subspace

$$\mathscr{M}_n = \chi_{\mathbb{D}}\mathscr{N}_n + L^2_a(G). \tag{1.2}$$

One shows that $\mathscr{M}_n$ is closed in $L^2_a(G\backslash\mathbb{T})$ and that

$$\dim(\mathscr{M}_n/z\mathscr{M}_n) = \dim(\mathscr{N}_n/z\mathscr{N}_n) + \dim(L^2_a(G)/zL^2_a(G)) = n + 1,$$

making $\mathscr{M}_n$ difficult to understand. By adding the condition $\chi_{\mathbb{D}} \in \mathscr{M}$, we avoid such pathologies as $\mathscr{M}_n$.

2. Preliminaries.

2.1. Sobolev and Besov spaces. Throughout this paper, G will be a Jordan region in the complex plane $\mathbb{C}$ (We make this restriction on G to avoid needless technicalities), $\mathbb{D} = \{z : |z| < 1\}$, and $\mathbb{T} = \{z : |z| = 1\}$. For the moment, we let $1 < p < 2$ and q be the conjugate index to p (so $q > 2$). The dual of $L^p(G) = L^p(G, dA)$, where dA is area measure, will be identified with $L^q(G)$ via the bilinear pairing

$$\langle f, g\rangle = \int_G fg\, dA. \tag{2.1}$$

Define the Sobolev space $W^{q,0}_1(G)$ as the closure of $C^\infty_0(G)$ (infinitely differentiable functions with compact support in G) in the norm

$$\|f\|_q = \left(\int |\nabla f|^q\, dA\right)^{1/q}.$$

Since $q > 2$, the Sobolev imbedding theorem yields that $W^{q,0}_1(G)$ is a Banach algebra of continuous functions [1], p. 115. Here we mean that every function has a continuous representative. The following describes $W^{q,0}_1(G\backslash E)$, where E is closed, in terms of zero sets. We refer the reader to [3] for a proof.

Proposition 2.1. *For $q > 2$, $W^{q,0}_1(G\backslash E) = \{f \in W^{q,0}_1(G) : f|_E = 0\}$.*

This next result of Havin [12] will be used later and relates the Bergman and Sobolev spaces.

Lemma 2.2. **(Havin)** *Let U be a bounded open set and $1 < p < \infty$. Then $f \in L^q(U)$ satisfies*

$$\int_U uf \, dA = 0 \ \ \forall u \in L^p_a(U)$$

if and only if there is an $F \in W_1^{q,0}(U)$ with $\bar{\partial} F = f$.

Define the *Besov space* B_q as the space of functions f on $\mathbb{T}$ with finite norm

$$\|f\|_{B_q} = \|f\|_{L^q(\mathbb{T},|d\zeta|)} + \Big(\int_{\mathbb{T}} \int_{\mathbb{T}} \Big| \frac{f(\zeta) - f(\xi)}{\zeta - \xi} \Big|^q \, |d\zeta||d\xi| \Big)^{1/q},$$

and note (since $q > 2$) that B_q can be continuously embedded into $\mathrm{Lip}_{1-2/q}(\mathbb{T})$ and hence B_q is a Banach algebra of continuous functions on $\mathbb{T}$ [5]. Define the *analytic Besov space*

$$AB_q = B_q \cap H^q,$$

where H^q is the usual Hardy space.

Remark. It is known [17] (see [28], Chapter 5, Section 5) that the analytic extension of $f \in AB_q$, given by the Poisson kernel, belongs to the L^q-Dirichlet space D_q of analytic functions on $\mathbb{D}$ with

$$\|f\|_{D_q} = \|f\|_{L^q(\mathbb{T},|d\zeta|)} + \left(\int_{\mathbb{D}} |f'(z)|^q \, dA(z) \right)^{1/q} < \infty$$

and moreover, the boundary values of $f \in D_q$ on $\mathbb{T}$ belongs to AB_q with the L^q-Dirichlet norm equivalent to the Besov norm. Thus we may identify $f(\zeta) \in AB_q$ with its analytic extension $f(z) \in D_q$.

The spaces $W_1^{q,0}(G)$ and B_q are related through restriction and extension. By standard trace theory [16], p. 182, [18], the trace operator

$$T : W_1^{q,0}(G) \to B_q, \ \ (Th)(\zeta) = h(\zeta)$$

is a well defined, continuous, surjective linear operator with, by Proposition 1. , $\ker(T) = W_1^{q,0}(G \backslash \mathbb{T})$. (Note that $h(\zeta)$ is well defined since $q > 2$ and so $W_1^{q,0}(G)$ is a space of continuous functions.) Thus T will induce the continuous invertible operator

$$\tilde{T} : W_1^{q,0}(G)/W_1^{q,0}(G \backslash \mathbb{T}) \to B_q, \ \ \tilde{T}[h](\zeta) = h(\zeta), \tag{2.2}$$

where $[h]$ is a coset of $W_1^{q,0}(G)/W_1^{q,0}(G\backslash\mathbb{T})$. By the above remark

$$\tilde{T}^{-1}(AB_q) = \{f \in W_1^{q,0}(G) : f|_{\mathbb{T}} \in AB_q\}/W_1^{q,0}(G\backslash\mathbb{T}). \tag{2.3}$$

2.2. Ideals of the Besov space. Since AB_q ($q > 2$) is a Banach algebra and analytic polynomials are dense, the ζ-invariant subspaces are precisely the ideals of AB_q and have been characterized by Shirokov [26] as follows:

Theorem 2.3. *If $\mathscr{F}$ is closed ideal of AB_q, there is a closed set $E \subset \mathbb{T}$ and an inner function I with*

$$\mathscr{F} = \mathscr{F}(E, I) \equiv \{f \in AB_q : f|_E = 0,\ f/I \in H^\infty\}.$$

Remark.

(i) The set E is the common zeros of $\mathscr{F}$ and the inner function I is the greatest common divisor of the inner parts of the functions in $\mathscr{F}$ [15], p. 85.

(ii) If $f \in AB_q$ with $f/I \in H^\infty$, then $f/I \in AB_q$ with $\|f/I\|_{B_q} \le C\|f\|_{B_q}$. In fact, division by the inner factor is a continuous operator on other spaces of 'smooth' functions on the disk [13].

(iii) From basic Hardy space theory [15], Chapter 5, an inner function I can be factored as $I = BS_\mu$, where B is a Blaschke product with zeros $\{a_k\} \subset \mathbb{D}$ (repeated according to multiplicity) and S_μ is a singular inner function with positive singular measure μ. Moreover [15], p. 68–69, S_μ cannot be continuously extended from $\mathbb{D}$ to any point in the support of μ. Hence, if $f \in AB_q$ with $f/I \in AB_q$, then f must vanish on the support of μ as well as the closure of $\{a_k\}$, i.e. on spec(I).

For a general closed set $E \subset \mathbb{T}$ and inner function I, the ideal $\mathscr{F}(E, I)$ might be zero. To understand when this happens, we make the following definition: A closed set $E \subset \mathbb{T}$ is called a *Carleson set* if

$$\int_{\mathbb{T}} \log \operatorname{dist}(\zeta, E)|d\zeta| > -\infty.$$

It is clear from the above condition that E has Lebesgue measure zero and it is known [7] that a E is a Carleson set if and only if E has Lebesgue measure zero and

$$\sum_n |I_n| \log |I_n| > -\infty,$$

where $\{I_n\}$ are the complimentary arcs of E.

Proposition 2.4. *The ideal $\mathscr{F}(E, I)$ is non-zero if and only if condition* (1.1) *is satisfied.*

Proof. The proof is essentially known (and true for other ideals of analytic functions [29] [30]) but not explicitly stated for the Besov space, so we outline it here. If condition (1.1) is satisfied, then we can find a non-zero $\phi \in A^\infty$ ($f^{(n)}$ analytic on $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}$ $\forall n \in \mathbb{N} \cup \{0\}$) with $\phi^{-1}(0) \cap \mathbb{T} = E$ and $\phi/B \in A^\infty$ [29] [30], Theorem 4.1. Notice that condition (1.1) implies that the support of μ is a Carleson set and we thus can apply [30], Corollary 4.8, to obtain a non-zero $\psi \in A^\infty$ with $\psi/S_\mu \in A^\infty$. Thus $0 \not\equiv \phi\psi \in \mathscr{F}(E, BS_\mu)$.

For the converse, it is known that if $f \in AB_q$ ($q > 2$), then f satisfies the Lipschitz condition [1], p. 97–98,

$$|f(z) - f(w)| \leq C|z - w|^{1-2/q} \quad z,\ w \in \bar{\mathbb{D}}. \tag{2.4}$$

In particular, if f is a non-zero element of $\mathscr{F}(E, BS_\mu)$, then by Jensen's inequality [15], p. 51–52,

$$\int_{\mathbb{T}} \log|f(\zeta)|\,|d\zeta| > -\infty$$

and by (2.4), along with the fact that f vanishes on $E \cup \operatorname{spec}(I)$,

$$\log|f(\zeta)| \leq (1 - 2/q) \log \operatorname{dist}(\zeta, E \cup \operatorname{spec}(I)) + C.$$

Hence condition (1.1) must be satisfied. □

3. The correspondence. We now relate our invariant subspaces of the Bergman space with the invariant subspaces of the Besov space. If $\chi_G \in \mathscr{M} \subset L^p_a(G\backslash\mathbb{T})$ is invariant, then $\mathscr{M}$ contains the polynomials and hence (since G is a Jordan region and polynomials are dense in $L^p_a(G)$)

$$L^p_a(G) \subset \mathscr{M} \subset L^p_a(G\backslash\mathbb{T}).$$

Thus,

$$L^p_a(G\backslash\mathbb{T})^\perp \subset \mathscr{M}^\perp \subset L^p_a(G)^\perp$$

with, by our bilinear pairing (2.1), $z\mathscr{M}^\perp \subset \mathscr{M}^\perp$. Here for a set $X \subset L^p(G)$ we let

$$X^\perp = \{g \in L^q(G) : \langle f, g\rangle = 0 \ \ \forall f \in X\}.$$

Thus, there is a one-to-one correspondence between the invariant subspaces

$$\chi_G \in \mathscr{M} \subset L^p_a(G\backslash\mathbb{T})$$

and the R-invariant subspaces of the quotient space

$$L^p_a(G)^{\perp}/L^p_a(G\backslash\mathbb{T})^{\perp}, \quad R[g]=[zg].$$

Our first result (which is also found in [2]) says that R is similar to M_ζ (multiplication by ζ) on the Besov space B_q. We will include a proof here so we can refer to parts of it later.

Theorem 3.1. *The linear transformation*

$$J: L^p_a(G)^{\perp}/L^p_a(G\backslash\mathbb{T})^{\perp} \to B_q$$

defined by

$$J[g](\zeta) = -\frac{1}{\pi}\int_G \frac{g(z)}{z-\zeta}\, dA(z)$$

is a continuous invertible operator with $JR = M_\zeta J$. Thus there is a one-to-one correspondence between the invariant subspaces $\chi_G \in \mathcal{M} \subset L^p_a(G\backslash\mathbb{T})$ and the lattice of ζ-invariant subspaces of B_q.

Proof. For any bounded open set U, we can apply Havin's Lemma (Lemma 2.2) and the Calderon-Zygmund theory [3], p. 266, to get that the operator

$$\bar{\partial}: W^{q,0}_1(U) \to L^p_a(U)^{\perp}$$

is continuous and invertible with inverse given by the Cauchy transform

$$(\bar{\partial}^{-1}g)(w) = (Cg)(w) = -\frac{1}{\pi}\int_U \frac{g(z)}{z-w}\, dA(z). \tag{3.1}$$

If R_z is multiplication by z on $L^p_a(U)^{\perp}$ and M_z is multiplication by z on $W^{q,0}_1(U)$ (both well defined and continuous) then, noticing that $\bar{\partial}(zf) = z\bar{\partial}f$ for all $f \in W^{q,0}_1(U)$, we have

$$R_z\bar{\partial} = \bar{\partial}M_z. \tag{3.2}$$

The Cauchy transform $C = \bar{\partial}^{-1}$ will induce the continuous invertible operator

$$\tilde{C}: L^p_a(G)^{\perp}/L^p_a(G\backslash\mathbb{T})^{\perp} \to W^{q,0}_1(G)/W^{q,0}_1(G\backslash\mathbb{T}).$$

Notice that R_z and M_z will induce the multiplication operators R and M on the cosets of $L^p_a(G)^{\perp}/L^p_a(G\backslash\mathbb{T})^{\perp}$ and $W^{q,0}_1(G)/W^{q,0}_1(G\backslash\mathbb{T})$ respectively with, by (3.2),

$$\tilde{C}R = M\tilde{C}.$$

Thus if we define

$$J : L^p_a(G)^\perp / L^p_a(G\backslash\mathbb{T})^\perp \to B_q$$

by $J = \tilde{T} \circ \tilde{C}$ (recall the definition of $\tilde{T}$ in (2.2)), we obtain

$$J[g](\zeta) = -\frac{1}{\pi}\int_G \frac{g(z)}{z-\zeta}\, dA(z)$$

and $JR = M_\zeta J$, where M_ζ is multiplication by ζ on B_q. □

Corollary 3.2. *If $\chi_{\mathbb{D}}, \chi_G \in \mathcal{M} \subset L^p_a(G\backslash\mathbb{T})$ is invariant and $g \in \mathcal{M}^\perp$, then*

$$J[g](\zeta) = -\frac{1}{\pi}\int_{G\backslash\mathbb{D}} \frac{g(z)}{z-\zeta} dA(z).$$

The function $J[g] \in AB_q$ and hence there is a one-to-one correspondence between the invariant subspaces $\chi_{\mathbb{D}}$, $\chi_G \in \mathcal{M}$ and the ideals of the analytic Besov space AB_q.

Proof. If $\chi_G, \chi_{\mathbb{D}} \in \mathcal{M} \subset L^p_a(G\backslash\mathbb{T})$, then by the invariance of $\mathcal{M}$, we get

$$L^p_a(G) \bigvee \chi_{\mathbb{D}} L^p_a(\mathbb{D}) \subset \mathcal{M} \subset L^p_a(G\backslash\mathbb{T}).$$

Taking annihilators one obtains

$$L^p_a(G\backslash\mathbb{T})^\perp \subset \mathcal{M}^\perp \subset L^p_a(G)^\perp \cap (\chi_{\mathbb{D}} L^p_a(\mathbb{D}))^\perp.$$

Thus if $g \in \mathcal{M}^\perp$, then $Cg \in W^{q,0}_1(G)$ and since $\mathcal{M}^\perp \subset (\chi_{\mathbb{D}} L^p_a(\mathbb{D}))^\perp$, we have

$$0 = \langle g, \chi_{\mathbb{D}}(z-\lambda)^{-1}\rangle = \int_{\mathbb{D}} \frac{g}{z-\lambda} dA$$

for all $|\lambda| \geq 1$ (Note that $(z-\lambda)^{-1} \in L^p_a(\mathbb{D})$ for $1 < p < 2$). Thus

$$J[g](\zeta) = (TCg)(\zeta) = -\frac{1}{\pi}\int_{G\backslash\mathbb{D}} \frac{g}{z-\zeta}\, dA.$$

The above function belongs to B_q and is analytic on $\mathbb{D}$, hence $J[g] \in AB_q$. □

Notation. If $\mathscr{F}$ is an ideal of AB_q we let $\mathscr{M}_{\mathscr{F}}$ be the unique invariant subspace of $L^p_a(G\backslash\mathbb{T})$ which contains χ_G and $\chi_{\mathbb{D}}$ and corresponds to $\mathscr{F}$ via J. One checks that

$$\mathscr{M}_{\mathscr{F}} = (\bar{\partial}\{f \in W^{q,0}_1(G) : f|_{\mathbb{T}} \in \mathscr{F}\})^{\perp}. \tag{3.3}$$

4. Invariant subspaces of Bergman spaces. Before proceeding to our main results, we first make a comment about inner functions. If I is an inner function and $I = BS_\mu$, then I is analytic for all points in the complex plane with the exception of the support of μ, $\{1/\bar{a_k}\}$, and the accumulation points of $\{a_k\}$. Moreover if $w^* = 1/\bar{w}$, then for $|z| > 1$, $I(z) = I(z^*)^*$. Thus $|I(z)|^{-1} \leq 1$ for all $|z| > 1$ and so $\chi_{G\backslash\mathbb{D}}/I \in L^p_a(G\backslash\mathbb{T})$. We also define $V(I)$ to be the set of inner functions ϕ which divide I, that is $I/\phi \in H^\infty$.

Theorem 4.1. *Let $1 < p < 2$ and $\chi_G, \chi_{\mathbb{D}} \in \mathscr{M} \subset L^p_a(G\backslash\mathbb{T})$ be invariant. Then there is a closed set $E \subset \mathbb{T}$ and an inner function I with*

$$\mathscr{M} = \mathscr{M}_{\mathscr{F}(E,I)} = L^p_a(G\backslash E) \bigvee \{\frac{\chi_{G\backslash\mathbb{D}}}{\phi} L^p_a(G) : \phi \in V(I)\}$$

Moreover $\mathscr{M} \neq L^p_a(G\backslash\mathbb{T})$ if and only if condition (1.1) *is satisfied.*

Proof. By (3.3), the unique subspace $\mathscr{M}_{\mathscr{F}(E,I)}$ corresponding to $\mathscr{F}(E,I)$ is

$$\mathscr{M}_{\mathscr{F}(E,I)} = (\bar{\partial}\{f \in W^{q,0}_1(G) : f|_{\mathbb{T}} \in \mathscr{F}(E,I)\})^{\perp}.$$

To finish, it suffices to show

$$\begin{aligned} &(\bar{\partial}\{f \in W^{q,0}_1(G) : f|_{\mathbb{T}} \in \mathscr{F}(E,I)\})^{\perp} \\ = \;& L^p_a(G\backslash E) \bigvee \left\{\frac{\chi_{G\backslash\mathbb{D}}}{\phi} L^p_a(G) : \phi \in V(I)\right\}. \end{aligned}$$

Let $h \in L^p_a(G\backslash E)$ and $f \in W^{q,0}_1(G)$ with $f|_{\mathbb{T}} \in \mathscr{F}(E,I)$. Then by Proposition 2.1, $f \in W^{q,0}_1(G\backslash E)$ and so by Havin's Lemma, Lemma 2.2,

$$\langle \bar{\partial} f, h\rangle = 0.$$

Thus by (3.3), $L^p_a(G\backslash E) \subset \mathscr{M}_{\mathscr{F}(E,I)}$.

Let $f \in W^{q,0}_1(G)$ with $f|_{\mathbb{T}} \in \mathscr{F}(E,I)$. Then for $\phi \in V(I)$,

$$\int_G \bar{\partial} f \frac{\chi_{G\backslash\mathbb{D}}}{\phi}\, dA \;=\; \lim_{\varepsilon\to 0} \int_{G\backslash\{|z|<1+\varepsilon\}} \frac{\bar{\partial} f}{\phi}\, dA \;= \tag{4.1}$$

$$= \lim_{\varepsilon\to 0}\int_{G\backslash\{|z|<1+\varepsilon\}} \bar\partial\left(\frac{f}{\phi}\right)\, dA.$$

By Green's theorem [31], p. 54, and the fact that $f = 0$ on the boundary of G, this becomes

$$-\lim_{\varepsilon\to 0}\frac{1}{2i}\int_{|z|=1+\varepsilon}\frac{f}{\phi}dz.$$

By the Lebesgue dominated convergence theorem and the fact that $f|_{\mathbb{T}}/\phi \in H^\infty$ we obtain

$$-\frac{1}{2i}\int_{\mathbb{T}}\frac{f}{\phi}dz = 0.$$

By (3.3) we get

$$\frac{\chi_{G\backslash\mathbb{D}}}{\phi} \in \mathscr{M}_{\mathscr{F}(E,I)}.$$

Using the invariance of $\mathscr{M}_{\mathscr{F}(E,I)}$ and the density of polynomials in $L^p_a(G)$, we have

$$L^p_a(G\backslash E)\bigvee\left\{\frac{\chi_{G\backslash\mathbb{D}}}{\phi}L^p_a(G) : \phi\in V(I)\right\}\subset \mathscr{M}_{\mathscr{F}(E,I)}. \tag{4.2}$$

Taking annihilators and then $C = \bar\partial^{-1}$ of (4.2) will yield

$$C\mathscr{M}^{\perp}_{\mathscr{F}(E,I)} \subset C\left(L^p_a(G\backslash E)^{\perp}\bigcap\left\{\left(\frac{\chi_{G\backslash\mathbb{D}}}{\phi}L^p_a(G)\right)^{\perp} : \phi\in V(I)\right\}\right). \tag{4.3}$$

To prove equality in (4.3) and thus finish the proof, we let g belong to the right hand side of (4.3) and show that $g \in C\mathscr{M}^{\perp}_{\mathscr{F}(E,I)}$ by showing $g|_{\mathbb{T}} \in \mathscr{F}(E,I)$, see (3.3). To do this, notice from (4.3) that $g = C(\bar\partial g)$ with

$$0 = -\frac{1}{\pi}\int_G \frac{\bar\partial g}{z-\lambda}\, dA = g(\lambda) \quad \forall\lambda\in E \tag{4.4}$$

$$\int_{G\backslash\mathbb{D}}\frac{\bar\partial g}{I}z^n\, dA = 0 \quad \forall n\in\mathbb{N}\cup\{0\}. \tag{4.5}$$

By (4.5),

$$0 \;=\; \int_{G\backslash\mathbb{D}}\frac{\bar\partial g}{I}z^n\, dA \;=$$

$$\begin{aligned} &= \lim_{\varepsilon\to 0}\int_{G\backslash\{|z|<1+\varepsilon\}} \frac{\bar{\partial} g}{I} z^n \, dA \\ &= \lim_{\varepsilon\to 0}\int_{G\backslash\{|z|<1+\varepsilon\}} \bar{\partial}\left(\frac{g}{I} z^n\right) \, dA \quad \forall n \in \mathbb{N}\cup\{0\}, \end{aligned}$$

which, by Green's Theorem [31], p. 54, becomes

$$0 = -\lim_{\varepsilon\to 0}\frac{1}{2i}\int_{|z|=1+\varepsilon} \frac{g}{I} z^n dz \quad \forall n \in \mathbb{N}\cup\{0\}.$$

By the Lebesgue dominated convergence theorem we obtain

$$0 = -\frac{1}{2i}\int_{\mathbb{T}} \frac{g}{I} z^n dz \quad \forall n \in \mathbb{N}\cup\{0\},$$

which, by the F. and M. Riesz theorem [15], p. 47, yields $g|_{\mathbb{T}}/I \in H^\infty$. Thus $g|_{\mathbb{T}}/I$ belongs to $\mathscr{F}(E,I)$ and we are done.

Finally, notice that from (3.3) and Proposition 2.1 that $\mathscr{F}(E,I) = 0$ if and only if $\mathscr{M}_{\mathscr{F}(E,I)} = L^p_a(G\backslash\mathbb{T})$. So from Proposition 2.4, $\mathscr{M}_{\mathscr{F}(E,I)} \neq L^p_a(G\backslash\mathbb{T})$ if and only if condition (1.1) is satisfied. □

We say that an inner function ϕ is a *multiple* of an inner function ψ if $\phi/\psi \in H^\infty$. We say an inner function I is the *least common multiple* of the inner functions I_1 and I_2 if I is a multiple of I_1 and I_2 and if the inner function ϕ is a multiple of I_1 and I_2, then ϕ is a multiple of I.

Corollary 4.2. *If the inner function I is the least common multiple of the inner functions I_1 and I_2, then*

$$\mathscr{M}_{\mathscr{F}(E,I)} = L^p_a(G\backslash E) \bigvee \frac{\chi_{G\backslash\mathbb{D}}}{I_1} L^p_a(G) \bigvee \frac{\chi_{G\backslash\mathbb{D}}}{I_2} L^p_a(G).$$

Proof. As in the proof above, we need to show

$$\begin{aligned} &(\bar{\partial}\{f \in W^{q,0}_1(G) : f|_{\mathbb{T}} \in \mathscr{F}(E,I)\})^\perp \\ &= L^p_a(G\backslash E) \bigvee \frac{\chi_{G\backslash\mathbb{D}}}{I_1} L^p_a(G) \bigvee \frac{\chi_{G\backslash\mathbb{D}}}{I_2} L^p_a(G). \end{aligned}$$

Using the same proof as above, one shows

$$L^p_a(G\backslash E) \subset (\bar{\partial}\{f \in W^{q,0}_1(G) : f|_{\mathbb{T}} \in \mathscr{F}(E,I)\})^\perp = \mathscr{M}_{\mathscr{F}(E,I)}. \tag{4.6}$$

Letting $f \in W_1^{q,0}(G)$ with $f|_{\mathbb{T}} \in \mathscr{F}(E,I)$, we see that $f|_{\mathbb{T}}/I_1$ and $f|_{\mathbb{T}}/I_2$ belong to H^∞, and thus by (4.1)

$$\frac{\chi_{G\setminus\mathbb{D}}}{I_1} \quad \text{and} \quad \frac{\chi_{G\setminus\mathbb{D}}}{I_2}$$

belong to $\mathscr{M}_{\mathscr{F}(E,I)}$. Now apply (4.6), the invariance of $\mathscr{M}_{\mathscr{F}(E,I)}$, the density of polynomials in $L_a^p(G)$, to show

$$\mathscr{M}_{\mathscr{F}(E,I)} \supset L_a^p(G\setminus E) \bigvee \frac{\chi_{G\setminus\mathbb{D}}}{I_1} L_a^p(G) \bigvee \frac{\chi_{G\setminus\mathbb{D}}}{I_2} L_a^p(G). \tag{4.7}$$

Taking annihilators and then $C = \bar{\partial}^{-1}$ of (4.7) will yield

$$C\mathscr{M}^{\perp}_{\mathscr{F}(E,I)} \subset W_1^{q,0}(G\setminus E) \cap C\left(\left(\frac{\chi_{G\setminus\mathbb{D}}}{I_1} L_a^p(G)\right)^{\perp} \cap \left(\frac{\chi_{G\setminus\mathbb{D}}}{I_2} L_a^p(G)\right)^{\perp}\right). \tag{4.8}$$

To prove equality in (4.8), and finish the proof, we let g belong to the right hand side of (4.8) and show $g \in C\mathscr{M}^{\perp}_{\mathscr{F}(E,I)}$ by showing $g|_{\mathbb{T}} \in \mathscr{F}(E,I)$. From (4.8), $g = C(\bar{\partial} g)$ with

$$-\frac{1}{\pi}\int_G \frac{\bar{\partial} g}{z-\lambda}\, dA = g(\lambda) = 0 \quad \forall \lambda \in E,$$

$$\int_{G\setminus\mathbb{D}} \frac{\bar{\partial} g}{I_1} z^n\, dA = 0 \quad \forall n \in \mathbb{N} \cup \{0\},$$

$$\int_{G\setminus\mathbb{D}} \frac{\bar{\partial} g}{I_2} z^n\, dA = 0 \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Proceed as in the proof above (using the F. and M. Riesz Theorem) to show that $g|_{\mathbb{T}}/I_1$ and $g|_{\mathbb{T}}/I_2$ belong to H^∞ (Just replace the inner function I in (4.5) with I_1 and I_2 respectively.). Letting I_g be the inner part of $g|_{\mathbb{T}}$, we see that I_g is a multiple of both I_1 and I_2. Since I is the least common multiple of I_1 and I_2, then I_g must be a multiple of I, making $g/I \in H^\infty$. Thus $g|_{\mathbb{T}} \in \mathscr{F}(E,I)$ and we are done. □

5. The classical Dirichlet space. For $q > 2$, every non-zero subspace of AB_q is an ideal of the form

$$IH^\infty \cap \{f \in AB_q : f|_E = 0\}$$

for some inner function I and Carleson set E. For $q = 2$, AB_2 is a well studied space of analytic functions which is not an algebra and whose invariant subspaces are not completely understood. AB_2 is better known as the Dirichlet space D_2 and are the radial limit functions of analytic $f(z)$ on $\mathbb{D}$ with finite Dirichlet integral

$$\int_{\mathbb{D}} |f'(z)|^2 \, dA(z).$$

If $f \in AB_2$, a theorem of Beurling [6] says that the radial limit

$$\lim_{r \to 1} f(r\zeta)$$

exists everywhere except possibly for a set of Bessel capacity C_2 (see below) zero. For $f \in AB_2$, let

$$Z(f) = \{\zeta \in \mathbb{T} : \lim_{r \to 1} f(r\zeta) = 0\}$$

and for a set $E \subset \mathbb{T}$, let

$$AB_{2,E} = \{f \in AB_2 : C_2(E \backslash Z(f)) = 0\}.$$

$AB_{2,E}$ is a closed subspace of AB_2 [7] as is

$$\mathscr{F}_{E,I} \equiv IH^2 \cap AB_{2,E},$$

for an inner function I and a set $E \subset \mathbb{T}$, and it is a conjecture that every invariant subspace of the Dirichlet space has this form.

We will show, in a similar way to the $1 < p < 2$ case, that the invariant subspaces of AB_2 are in one-to-one correspondence with the invariant subspaces

$$\chi_{\mathbb{D}}, \chi_G \in \mathscr{M} \subset L^2_a(G \backslash \mathbb{T})$$

and then identify $\mathscr{M}_{\mathscr{F}_{E,I}}$. To proceed, we must first take care of some technical matters.

5.1. Capacity. Following [3], we define the C_2-*capacity* of a compact set F by

$$C_2(F) = \inf \int |\nabla \phi|^2 \, dA,$$

where the inf is taken over all real-valued functions $\phi \in C_0^\infty$ with $\phi = 1$ on F. We extend this definition to arbitrary sets E by

$$C_2(E) = \sup\{C_2(F) : F \subset E, \quad F \text{ compact}\}$$

and define the *exterior capacity* $C_2^*(E)$ of an arbitrary set E by

$$C_2^*(E) = \inf\{C_2(G) : G \supset E, \quad G \text{ open}\}.$$

A set E is said to *capacitable* if $C_2(E) = C_2^*(E)$.

Remark. The Sobolev space W_1^2 is equal to L_1^2, the space of Bessel potentials, and thus the Bessel capacity is equivalent to C_2^* [14]. We bring this to the readers attention since the literature often uses both definitions of capacity.

One notes [3] that C_2^* is a monotone, subadditive set function and that the Borel sets are capacitable. We say a set E is *quasi-closed* of given $\varepsilon > 0$, there is an open set W with $C_2(W) < \varepsilon$ and $E \backslash W$ is closed. One argues, using the fact that Borel sets are capacitable, that a quasi-closed set is capacitable, as is the difference of any two quasi-closed sets. As mentioned in the introduction, we say a property holds *quasi-everywhere* if the set for which it fails has exterior capacity zero.

Since functions in $W_1^{2,0}(G)$ are not always continuous (or even bounded), we introduce a suitable substitution for continuity. A complex-valued function f is *quasi-continuous* if for every $\varepsilon > 0$ there is an open set W with $C_2(W) < \varepsilon$ and $f|_{\mathbb{C}\backslash W}$ continuous. One can show [3], Lemma 1, Theorem 2, that every $f \in W_1^{2,0}(G)$ has a quasi-continuous representative and in fact, one can find a formula for the quasi-continuous representative of a Sobolev function. For $f \in W_1^{2,0}(G)$ we define

$$f^*(w) = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{|z-w|<r} f(z)\, dA(z) \tag{5.1}$$

whenever this limit exists and notice by the Lebesgue differentiation theorem, $f = f^*$ a.e. By [11], $f^*(w)$ is defined quasi-everywhere and moreover f^* is quasi-continuous. This next result of Bagby [3], Theorem 4, describes $W_1^{2,0}(G \backslash E)$, E closed, in terms of zero sets.

Proposition 5.1. $W_1^{2,0}(G \backslash E) = \{f \in W_1^{2,0}(G) : f^*|_E = 0 \text{ } q.e.\}$.

5.2. Traces. A result of [16], p. 182, shows that the trace operator

$$T : W_1^{2,0}(G) \to B_2, \quad Tf = f^*|_{\mathbb{T}}$$

is a well defined, continuous, surjective operator with, by Proposition 5.1,

$$\ker(T) = W_1^{2,0}(G \backslash \mathbb{T}).$$

Remark. By [18], one can also define the (continuous) trace operator

$$\mathrm{tr} : W_1^{2,0}(G) \to B_2$$

$$(\mathrm{tr} f)(\zeta) = \lim_{r \uparrow 1} f(r\zeta)$$

and notice that this limit exists a.e. $|d\zeta|$ and in $L^2(\mathbb{T}, |d\zeta|)$. Also notice that $T(\phi) = \mathrm{tr}(\phi)$ for all $\phi \in C_0^\infty(G)$. Thus $Tf = \mathrm{tr} f$ a.e. $|d\zeta|$ for all $f \in W_1^{2,0}(G)$.

Thus (as before) T will induce

$$\tilde{T} : W_1^{2,0}(G)/W_1^{2,0}(G \backslash \mathbb{T}) \to B_2, \ \ \tilde{T}[h] = h^*|_{\mathbb{T}}$$

and $\tilde{T}$ will be continuous and invertible. So (as before) we define

$$J : L_a^2(G)^\perp / L_a^2(G \backslash \mathbb{T})^\perp \to B_2$$

by $J = \tilde{T} \circ \tilde{C}$ to obtain

$$J[g](\zeta) = -\frac{1}{\pi} \int_G \frac{g(z)}{z - \zeta} dA(z) \quad \text{q.e.}$$

and $JR = M_\zeta J$ (see [2] for details). Thus there is a one-to-one correspondence between the invariant subspaces $\chi_G \in \mathscr{M} \subset L_a^2(G \backslash \mathbb{T})$ and the invariant subspaces of B_2. One also shows (as before) that for an invariant subspace $\chi_{\mathbb{D}}, \chi_G \in \mathscr{M} \subset L_a^2(G \backslash \mathbb{T})$ and $g \in \mathscr{M}^\perp$,

$$J[g](\zeta) = -\frac{1}{\pi} \int_{G \backslash \mathbb{D}} \frac{g(z)}{z - \zeta} \, dA(z)$$

and so $J[g] \in AB_2$. So, as before, there is a one-to-one correspondence between the invariant subspaces $\chi_{\mathbb{D}}, \chi_G \in \mathscr{M}$ and the invariant subspaces of AB_2.

5.3. Zero Sets. For a quasi-closed set $E \subset \mathbb{T}$, we can find a sequence of closed sets $F_1 \subset F_2 \subset \cdots \subset E$ with $C_2(F_n) \to C_2(E)$. Since $L_a^2(G \backslash F_n)$ increases with n, we can define the invariant subspace

$$\mathscr{M}(E) = \overline{\bigcup_n L_a^2(G \backslash F_n)}^{L^2}. \tag{5.2}$$

One can prove the following basic facts about $\mathscr{M}(E)$ [23]:

Proposition 5.2. *For quasi-closed sets $E, F \subset K$*

(1) $\mathscr{M}(E)$ *is independent of the choice of* $\{F_n\}$.
(2) $\mathscr{M}(E) \subset \mathscr{M}(F) \Leftrightarrow C_2(E\backslash F) = 0$.
(3) $\mathscr{M}(E) = \mathscr{M}(F) \Leftrightarrow C_2(E\Delta F) = 0$.

Remark. There are quasi-closed sets $E \subset \mathbb{T}$ for which $\mathscr{M}(E)$ cannot be written as $L^2_a(G\backslash F)$ for any closed $F \subset K$ [23], Proposition 4.3.

One also notes that

$$W_2(E) \equiv C(\mathscr{M}(E)^\perp) = \bigcap_n W_1^{2,0}(G\backslash F_n) \tag{5.3}$$

and by Proposition 2.1, $f \in W_1^{2,0}(G)$ belongs to $W_2(E)$ if and only if $f^* = 0$ quasi-everywhere on E. From this one has

$$\begin{aligned} J\left(\mathscr{M}(E)^\perp / L^2_a(G\backslash\mathbb{T})^\perp \right) &= T(C\mathscr{M}(E)^\perp) = T(W_2(E)) \\ &= \{f \in B_2 : f^*|_E = 0 \text{ q.e.}\}. \end{aligned} \tag{5.4}$$

Remark. The subspace $B_{2,E}(\mathbb{T}) \equiv \{f \in B_2 : f^*|_E = 0 \text{ q.e.}\}$ can be described in several equivalent ways. If $f \in B_2$, then (as mentioned in the introduction, see also [20]) the radial limit of its harmonic extension exists q.e. and is a quasi-continuous function on $\mathbb{T}$. Since f^* is also a quasi-continuous function on $\mathbb{T}$ which equals the radial limit function a.e. $|d\zeta|$, then [20], Proposition 2.1 (c),

$$f^*(\zeta) = \lim_{r\to 1} f(r\zeta) \quad \text{q.e.}$$

Thus we have that $Z(f)$ is quasi-closed (an easy exercise using the definition of quasi-continuity) and

$$B_{2,E}(\mathbb{T}) = \{f \in B_2 : C_2(E\backslash Z(f)) = 0\}$$

from which $AB_{2,E} = B_{2,E} \cap H^2$.

6. Invariant subspaces for $p = 2$. For an inner function I and a quasi-closed set $E \subset \mathbb{T}$, recall that

$$\mathscr{F}_{E,I} = IH^2 \cap AB_{2,E}$$

is a (closed) invariant subspace of AB_2. We now identify $\mathscr{M}_{\mathscr{F}_{E,I}}$.

Theorem 6.1.

$$\mathcal{M}_{\mathcal{F}_{E,I}} = \mathcal{M}(E) \bigvee \{ \frac{\chi_{G\backslash\mathbb{D}}}{\phi} L_a^2(G) : \phi \in V(I) \}.$$

Proof. The proof is nearly identical to the proof of Theorem 4.1, except for some technicalities. We first show that

$$\mathcal{M}(E) = \overline{\bigcup_n L_a^2(G\backslash F_n)}^{L^2} \subset \mathcal{M}_{\mathcal{F}_{E,I}}. \tag{6.1}$$

Fix n and let $h \in L_a^2(G\backslash F_n)$ and $f \in W_1^{2,0}(G)$ with $Tf \in IH^2 \cap B_{2,E}$. Then by Proposition 2.1, $f \in W_1^{2,0}(G\backslash F_n)$ and so by Havin's Lemma, Lemma 2.2,

$$\langle \bar{\partial} f, h \rangle = 0.$$

Thus by the definition of $\mathcal{M}(E)$, we have (6.1).

By (3.3), notice that

$$\mathcal{M}_{\mathcal{F}_{E,I}} = \left(\bar{\partial}\{ f \in W_1^{2,0}(G) : Tf \in IH^2 \cap AB_{2,E} \} \right)^{\perp}.$$

So let $f \in W_1^{2,0}(G)$ with $Tf = IH^2 \cap AB_{2,E}$. Then for all $\phi \in V(I)$,

$$\int_G \bar{\partial} f \frac{\chi_{G\backslash\mathbb{D}}}{\phi}\, dA = \int_{G\backslash\mathbb{D}} \frac{\bar{\partial} f}{\phi}\, dA = \lim_{\varepsilon \to 0} \int_{G\backslash\{|z|<1+\varepsilon\}} \bar{\partial}\left(\frac{f}{\phi}\right)\, dA,$$

which by Green's theorem [31], p. 54, equals

$$\lim_{\varepsilon \to 0} -\frac{1}{2i} \int_{|z|=1+\varepsilon} \frac{f}{\phi} dz.$$

By [18], $f((1+\varepsilon)\zeta) \to f(\zeta)$ in $L^2(\mathbb{T}, |d\zeta|)$. Thus using the fact that $Tf/\phi \in H^2$, the above becomes

$$-\frac{1}{2i} \int_{\mathbb{T}} \frac{Tf}{\phi} dz = 0,$$

hence $\chi_{G\backslash\mathbb{D}}/\phi \in \mathcal{M}_{\mathcal{F}_{E,I}}$. So, by the invariance of $\mathcal{M}_{\mathcal{F}_{E,I}}$, the density of polynomials in $L_a^2(G)$, and (6.1),

$$\mathcal{M}(E) \bigvee \left\{ \frac{\chi_{G\backslash\mathbb{D}}}{\phi} L_a^2(G) : \phi \in V(I) \right\} \subset \mathcal{M}_{\mathcal{F}_{E,I}}.$$

By taking annihilators and then $C = \bar{\partial}^{-1}$ we get (using (5.3))

$$C\mathcal{M}^{\perp}_{\mathcal{F}_{E,I}} \subset W_2(E) \cap C\left(\bigcap\left\{\left(\frac{\chi_{G\setminus\mathbb{D}}}{\phi}L^2_a(G)\right)^{\perp} : \phi \in V(I)\right\}\right).$$

Let $g = C(\bar{\partial} g)$ belong to the right hand side of the above. Then $g^*|_E = 0$ q.e. and

$$\int_{G\setminus\mathbb{D}} \bar{\partial} g \frac{z^n}{I}\, dA = 0 \quad \forall n \in \mathbb{N} \cup \{0\}.$$

As before (using Green's theorem)

$$\int_{\mathbb{T}} \frac{g}{I} z^n dz = 0 \quad \forall n \in \mathbb{N} \cup \{0\}$$

and so $(Tg)/I \in H^2$ which means $Tg \in IH^2 \cap AB_{2,E}$. Thus $g \in C\mathcal{M}^{\perp}_{\mathcal{F}_{E,I}}$ and we are done. □

Using a similar proof as in Corollary 4.2, one can prove the following corollary:

Corollary 6.2. *If the inner function I is the least common multiple of the inner functions I_1 and I_2, then*

$$\mathcal{M}_{\mathcal{F}_{E,I}} = \mathcal{M}(E) \bigvee \frac{\chi_{G\setminus\mathbb{D}}}{I_1} L^2_a(G) \bigvee \frac{\chi_{G\setminus\mathbb{D}}}{I_2} L^2_a(G).$$

As in the $1 < p < 2$ case we have $\mathcal{M}_{\mathcal{F}_{E,I}} = L^2_a(G\setminus\mathbb{T})$ if and only if $\mathcal{F}_{E,I} = 0$. However, understanding when $\mathcal{F}_{E,I}$ is non-trivial is more complicated and is yet unknown. In fact, understanding the zero sets for the Dirichlet space (i.e. when $\mathcal{F}_{E,1} \not\equiv 0$) remains an open problem [10]. For example, there are Blaschke products which divide Dirichlet functions and whose zeros accumulate near every point of $\mathbb{T}$, in stark contrast to the AB_q ($q > 2$) case where the zeros must accumulate on a Carleson set.

By [22], it is known that every invariant subspace $\mathcal{F}$ of AB_2 is of the form

$$\mathcal{F} = IH^2 \cap [f],$$

where I is an inner function, $f \in AB_2$ is outer and $[f] = \text{span}\{\zeta^n f : n = 0, 1, 2, \cdots\}$. Certainly $\mathcal{F} \subset \mathcal{F}_{Z(f),I}$ and so by (3.3)

$$\mathcal{M}_{\mathcal{F}} \supset \mathcal{M}_{\mathcal{F}_{Z(f),I}}.$$

It is a conjecture that $[f] = AB_{2,Z(f)}$ and hence we will have equality above, thus giving us a complete characterization of the invariant subspaces $\chi_{\mathbb{D}}, \chi_G \in \mathcal{M} \subset L^2_a(G\backslash\mathbb{T})$. For certain outer functions f, $[f] = AB_{2,Z(f)}$, but the general question still remains open.

7. $p > 2$. We mention that the results in the previous section have generalizations to $p > 2$. The techniques are exactly the same except for some technicalities which we mention now.

For $p \geq 2$ the appropriate space to look at is B_q and the capacity used is the C_q capacity (defined in an analogous way). The capacity theory is the same and the trace operator T is defined as before.

If $1 < q \leq 2$ and $f \in AB_q = D_q$, it is known [8] that the radial limit

$$\lim_{r \to 1} f(r\zeta)$$

exist everywhere except possibly on a set of q-Bessel capacity (equivalently the C_q^* capacity) zero. As is the $q = 2$ case, we define the set

$$Z(f) = \{\zeta \in \mathbb{T} : \lim_{r \to 1} f(r\zeta) = 0\}.$$

For a quasi-closed set (with respect to the C_q capacity) $E \subset \mathbb{T}$ we define

$$AB_{q,E} = \{f \in AB_q : C_q(E\backslash Z(f)) = 0\}.$$

A result of Carleson [7] says that $AB_{2,E}$ is a closed subspace of AB_2. This next result (which is known but we could not find a proof) says the same for $AB_{q,E}$.

Proposition 7.1. *For $1 < q \leq 2$ and a set $E \subset \mathbb{T}$, $AB_{q,E}$ is a closed subspace of AB_q.*

Proof. It is known (by observing that W^q_1 is the same as the space of Bessel potentials L^q_1) that if $\{f_i\}$ is a Cauchy sequence of quasi-continuous functions in $AB_q = D_q$ (i.e. quasi-everywhere defined on $\bar{\mathbb{D}}$ and quasi-continuous) then there is a quasi-continuous $f \in AB_q$ and a subsequence $f_{i_j} \to f$ quasi-everywhere (see [20], Proposition 2.1 for a proof in a slightly different setting).

Thus for $f \in AB_q$ we define $\tilde{f}$ q.e. on $\bar{\mathbb{D}}$ by setting $\tilde{f}$ to be $f(z)$ for $z \in \mathbb{D}$ and $\tilde{f}(\zeta)$ to be the the radial limit of f at ζ for $\zeta \in \mathbb{T}$. For $0 < r < 1$ define $f_r(z) = f(rz)$ and notice that $\tilde{f_r}$ is continuous on $\bar{\mathbb{D}}$ and $f_r \to f$ in L^q-Dirichlet norm. Thus by the above fact, $\tilde{f}$ is quasi-everywhere equal to a quasi-continuous function, making $\tilde{f}$ quasi-continuous on $\bar{\mathbb{D}}$.

So if $\{f_n\}$ is a Cauchy sequence in $AB_{q,E}$ then $\{\tilde{f}_n\}$ is a Cauchy sequence of quasi-continuous functions in $AB_{q,E}$ and by the above fact, the limit function must vanish q.e. on E. Thus $AB_{q,E}$ is closed. □

For $f \in AB_q$ we have that f^* and $\tilde{f}$ are quasi-continuous functions on $\bar{\mathbb{D}}$ with $f^* = \tilde{f}$ a.e. By [3], Theorem 2 (iii), $f^* = \tilde{f}$ q.e. As before, one defines $\mathcal{M}(E)$ and one has (using the above and (5.4)) that

$$J\left(\mathcal{M}(E)^{\perp}/L^p_a(G\backslash\mathbb{T})^{\perp} \right) = AB_{q,E}.$$

For an inner function I and a quasi-closed set $E \subset \mathbb{T}$ define

$$\mathcal{F}_{E,I} = IH^q \cap AB_{q,E}$$

and notice that this is a closed subspace of AB_q. Using the same proof as in the $q = 2$ case and the above, one proves that

$$\mathcal{M}_{\mathcal{F}_{E,I}} = \mathcal{M}(E) \bigvee \left\{ \frac{\chi_{G\backslash\mathbb{D}}}{\phi} L^p_a(G) : \phi \in V(I) \right\}.$$

One also proves, in exactly the same way as before, that if the inner function I is the least common multiple of the inner functions I_1 and I_2, then

$$\mathcal{M}_{\mathcal{F}_{E,I}} = \mathcal{M}(E) \bigvee \frac{\chi_{G\backslash\mathbb{D}}}{I_1} L^p_a(G) \bigvee \frac{\chi_{G\backslash\mathbb{D}}}{I_2} L^p_a(G).$$

8. Codimension. If $1 < p \leq 2$ and $S : L^p_a(G\backslash\mathbb{T}) \to L^p_a(G\backslash\mathbb{T})$ is $(Sf)(z) = zf(z)$, then for an invariant subspace $\mathcal{M}$ and $\lambda \in G\backslash\mathbb{T}$, $(S - \lambda)|_{\mathcal{M}}$ is a semi-Fredholm operator and

$$-\text{index}((S - \lambda)|_{\mathcal{M}}) = \dim(\mathcal{M}/(z - \lambda)\mathcal{M})$$

is constant on the components of $G\backslash\mathbb{T}$ [19], Lemma 2.1, and is called the *codimension* on the component of $G\backslash\mathbb{T}$. In [2] they prove the following formula:

$$\dim(\mathcal{M}/\mathcal{M}(z - \lambda)) = 1 + \dim(\mathcal{F}_{\mathcal{M}}/(\zeta - \lambda)\mathcal{F}_{\mathcal{M}}), \tag{8.1}$$

where $\mathcal{F}_{\mathcal{M}}$ is the unique invariant subspace of B_q that corresponds to $\mathcal{M}$ via the operator J.

Theorem 8.1. *If $1 < p \leq 2$ and $\chi_G, \chi_{\mathbb{D}} \in \mathcal{M} \subset L^p_a(G\backslash\mathbb{T})$ is a non-trivial invariant subspace, then for $\lambda \in G\backslash\mathbb{T}$,*

$$\dim(\mathcal{M}/(z - \lambda)\mathcal{M}) = \begin{cases} 1 & \textit{if } |\lambda| > 1 \\ 2 & \textit{if } |\lambda| < 1 \end{cases}$$

Proof. If $\lambda \in G\backslash\mathbb{T}$ with $|\lambda| > 1$, then $M_{\zeta-\lambda}$ is an invertible operator on $\mathscr{F}_{\mathscr{M}}$ and so $(\zeta - \lambda)\mathscr{F}_{\mathscr{M}} = \mathscr{F}_{\mathscr{M}}$. From (8.1) we get $\dim(\mathscr{M}/(z-\lambda)\mathscr{M}) = 1$.

Since $\mathscr{F}_{\mathscr{M}}$ is a non-trivial invariant subspace of AB_q then by a result of [19] $(q > 2)$ and [21] $(q = 2)$, $\dim(\mathscr{F}_{\mathscr{M}}/\zeta\mathscr{F}_{\mathscr{M}}) = 1$. Thus by (8.1), $\dim(\mathscr{M}/z\mathscr{M}) = 2$. Since the codimension is constant on the components of $G\backslash\mathbb{T}$ we are done. □

We mention that in general the invariant subspaces $\mathscr{F}$ of B_2 can be quite complicated, thus making the invariant subspaces $\chi_G \in \mathscr{M} \subset L^2_a(G\backslash\mathbb{T})$ difficult to describe. The invariant subspaces $\mathscr{F}$ with $\zeta\mathscr{F} = \mathscr{F}$ have been completely characterized [20] as $\mathscr{F} = B_{2,E}(\mathbb{T})$ for some quasi-closed $E \subset \mathbb{T}$ and thus by (5.4) $\mathscr{M}_{\mathscr{F}} = \mathscr{M}(E)$. The invariant subspaces $\mathscr{F}$ with $\zeta\mathscr{F} \neq \mathscr{F}$ (such subspaces are called *simply invariant*) are quite complicated. In fact, recall from the introduction that the invariant subspace $\mathscr{M}_n$ of the Bergman space, (1.2), has $\dim(\mathscr{M}_n/z\mathscr{M}_n) = n$. Thus by (8.1)

$$\dim(\mathscr{F}_{\mathscr{M}_n}/\zeta\mathscr{F}_{\mathscr{M}_n}) = n,$$

(see [24] for a specific example) which is in stark contrast to the analytic Dirichlet space AB_2 where the codimension of any non-zero invariant subspace is always one.

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