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A logical development of the real number system

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A LOGICAL DEVELOPMENT
OF THE
REAL NUMBER SYSTEM

BY

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PREFACE

This work is an axiomatic investigation of real numbers based on Edmund Landau's GRUNDLAGEN DER ANALYSIS. It endeavors, with the use of five postulates or axioms, to prove certain theorems used in elementary mathematics.

INTRODUCTION

Man and a few of the lower animals possess what may be called Number Sense. This is the power to recognise a change in number in a small collection without actually counting the collection.

Man as time passed began to give names to collections such as these: a flock of birds, a school of fish, a yoke of oxen. The process of putting one collection into one-to-one correspondence with another started man on his road to counting. After thousands of years he realized that a brace of pheasants and a yoke of oxen were alike in the fact that they both represented the number two. Man began to set up model collections which always denoted certain numbers. After generations of use, the name and sound of the collection had often degenerated and the new word that evolved beaame the symbol for the number, thus losing the significance of its connection with the original collection. The cardinal number is based on this principle of correspondence. We now no longer look for model cellections; we set one collection into one-to-one correspondence with the natural numbers; in other words, we count it.

Man has now learned to count. He has developed an ingenious counting board, abacus, to assist him in his calculations. Now he can add, subtract, multiply, and divide with the aid of his new instrument. But he has a problem, there is often an empty column, and when he sets down the result of his calculations, 49, 409, 4009, all look alike. An unknown Hindu represented this empty class by our symbol for zero called in Indian "sunya". Sunya was translated into Arabic as "sifr," meaning empty. This word passed into Latin as "cifra," and from Latin into English as "cipher." Cipher as it underwent these language changes collected a double meaning, it signified both zero and number. Finally the Italian zero was adopted in the sense that we use it today.

Fractions had different meanings in different number systems, the Greeks considering them as the ratio of two numbers, while the Egyptians considered them simply as parts of numbers. Our conception of a fraction is colored by both these ideas.

The Greeks had no conception of a negative number. The Hindus, about the seventh century A. D. were the first to use them.

Now we have integers, fractions, positive and negative numbers and zero making up the rational domain.

Pythagoras was the first to discover irrational numbers, when finding the length of the hypothenuse. Euclid's proof of the length of the diagonal of a square, involving irrational numbers, is based solely on number theory and has little to do with geometry.

Hilbert studied the postulates of Euclid's geometry and decided that corresponding postulates could be set up in arithmetic. Hilbert, Peacock, Boole and Peano were among the first to make an axiomatic investigation of the number system.

CONTENTS

CHAPTER IV. REAL NUMBERS

CHAPTER I

NATURAL NUMBERS

Section 1. Postulates

Let us consider the class N of natural numbers whose elements will be represented by the letters u, v, w, x, y, and z.

It is agreed that the concept of equality among the elements of N (symbol =), subject to the following rules is understood.

1) For any pair of numbers x, y, either

$$x = y$$

or

$$x \neq y$$

(x not equal to y).

2)

$$x = x$$

for all values of x.

3) If

$$x = y$$

then

$$y = x.$$

4) If

$$x = y, \; y = z$$

then

$$x = z.$$

The class N is defined by the following postulates.

Postulate 1: 1 is a natural number.

Postulate 2: For each x contained in N there is a consequent x' contained in N. If

$$x = y$$

then

$$x' = y'.$$

Postulate 3: 1 is not the consequent of any number.

$$x' \neq 1.$$

Postulate 4: If $x' = y'$

then

$$x = y.$$

Postulate 5 (Induction Postulate): Let there be a class M of natural numbers with the following properties.

I) 1 is contained in M.

II) If x is contained in M, then x' is contained in M.

Then M contains the class of all natural numbers.

Section 2. Addition.

<u>Theorem 1.2.1:</u> If $x \neq y$

then $x' \neq y'$.

Proof: Let us assume that

$$x' = y'.$$

Then by postulate 4 $x = y$

which is contrary to the hypothesis.

Therefore $x' \neq y'$.

<u>Theorem 1.2.2:</u> $x' \neq x$.

Proof: Let M be the class of x for which $x' \neq x$.

I) By postulate 1 and postulate 3

$$1' \neq 1.$$

Thus 1 is contained in M.

II) Assume that x is contained in M, then by theorem 1.2

$$(x')' \neq x'.$$

Therefore x' is contained in M, and hence by postulate 5, M is the class of all natural numbers N.

<u>Theorem 1.2.3:</u> If $x \neq 1$

then there is a u such that

$$x = u'.$$

Proof: Let M be that class of numbers containing 1 and those x's ($x \neq 1$) for which there is one such u.

I) 1 is contained in M.

II) If y is contained in M, then if $y = u$, $y' = u'$, hence y' is contained in M.

Therefore M contains the class of all natural numbers N.

<u>Theorem 1.2.4:</u> A number pair, x, y, can be combined in one and only one way to give $x + y$ (the sum), so that

1) $x + 1 = x'$ for all x,

2) $x + y' = (x + y)'$ for all x and all y.

Proof: A) We will show first that there is in the case of every fixed x at the most one possibility, $x + y$, for all y so defined, that

$$x + 1 = x',$$

and

$$x + y' = (x + y)'.$$

Let a_y and b_y be defined for all y, such that

$a_1 = x'$, $b_1 = x'$,

$a_{y'} = (a_y)'$, $b_{y'} = (b_y)'$.

Let M be the class of y for which

$$a_y = b_y.$$

I) $a_1 = x' = b_1$;

1 is contained in M.

II) For any y contained in M, we have $a_y = b_y$ by definition.

Then $(a_y)' = (b_y)'$ by postulate 2,

whence $a_{y'} = (a_y)' = (b_y)' = b_{y'}$,

and y' is in M.

The solution is unique, if it exists.

B) We shall now show that $x + y$ actually does exist, as defined by the following properties, for all x and y:

i) $x + 1 = x'$,

ii) $x + y' = (x + y)'$.

I) For $$x = 1,$$
define $$x + y = y',$$
then $$x + 1 = 1' = x',$$
and $$x + y' = (y')' = (x + y)'.$$

Therefore 1 is contained in M.

II) Now let x be any number in M. Then $x + y$ exists by assumption for all y. Define

$$x' + y = (x + y)'.$$

Then $$x' + 1 = (x + 1)' = (x')',$$
and $$x' + y' = (x + y')' = ((x + y)')' = (x' + y)',$$
hence x' belongs to M.

Therefore M is the set of all natural numbers, and the proof is complete.

<u>Theorem 1.2.5:</u> $$(x + y) + z = x + (y + z).$$

Proof: For a fixed x and y, let M be the class of z for which the assertion is true.

I) $$(x + y) + 1 = (x + y)' = x + y' = x + (y + 1);$$

and 1 is contained in M.

II) z is contained in M. Thus

$$(x + y) + z = x + (y + z),$$

then

$$(x + y) + z' = ((x + y) + z)' = (x + (y + z))'$$

$$x + (y + z)' = x + (y + z'),$$

then z' is contained in M.

The assertion holds for all z.

Theorem 1.2.6: $x + y = y + z.$

Proof: Let y be fixed, and let M be the class of x for which $x + y = y + x.$

I) Since $y + 1 = y',$

and by theorem 1.2.4 $1 + y = y',$

then $1 + y = y + 1$

and 1 is contained in M.

II) Since x is contained in M, and

$$x + y = y + x,$$

then $(x + y)' = (y + x)' = y + x'.$

By theorem 1.2.4 $x' + y = (x + y)',$

then $x' + y = y + x',$

therefore x' is in M, and M is the class of all natural numbers.

Theorem 1.2.7 $y \neq x + y$

Proof: For a fixed x let M be the class of y for which $y \neq x + y.$

I) By theorem 1.2.3 $1 \neq x'.$

By theorem 1.2.4 $1 \neq x + 1,$

then 1 is contained in M.

II) If y is in M, then

$$y \neq x + y,$$

and $$y' \neq (x + y)',$$

and $$y' \neq x + y',$$

hence y' is contained in M, and M is the class of all natural numbers N.

Theorem 1.2.8: If $$y \neq z$$

then $$x + y \neq y + z.$$

Proof: Let M be the class of x for which

$$x + y \neq y + z,$$

when y and z are fixed, with $y \neq z$.

I) $$y' \neq z',$$

then $$1 + y \neq 1 + z.$$

1 is contained in M

II) If x is contained in M, then

$$x + y \neq x + z,$$

and $$(x + y)' \neq (x + z)',$$

then $$x' + y \neq x' + z.$$

x' is contained in M, and M is the class of all natural numbers N.

Theorem 1.2.9: If x and y are given, then exactly one of the following cases holds.

1) $$x = y.$$

2) There is exactly one u such that

$$x = y - u.$$

3) There is exactly one v such that

$$y = x - v.$$

Proof: A) By theorem 1.2.7, 1) and 2), 1) and 3), and 2) and 3) are incompatible.

If, for example, 2) and 3) hold, then

$$x = y - u = (x - v) - u = x - (v - u) = (v - u) - x.$$

Therefore at the most one of 1), 2), or 3) holds.

B) If x is fixed, let M be the class of y for which one of cases 1), 2), or 3) holds.

I) For $y = 1$ according to theorem 1.2.3 either

$$x = 1 = y \quad \text{(case 1)}$$

or $$x = u' = 1 - u = y - u \quad \text{(case 2)},$$

then 1 is contained in M.

II) If y is contained in M, then if either

(case 1) for y) $\qquad x = y$

then $\qquad y' = y - 1 = x - 1 \quad$ (case 3) for y');

or (case 2) for y) $\qquad x = y - u$,

then when $\qquad u = 1$,

$$x = y - 1 = y' \quad \text{(case 1) for } y');$$

when $\qquad u \neq 1$

by theorem 1.2.3 $\qquad u = w' = 1 - w$,

$$x = y - (1 - w) = (y - 1) - w = y' - w \quad \text{(case 2) for } y');$$

or (case 3) for y) $\qquad y = x - v$, then $y' = (x + v)' = x + v'$ (case 3) for y').

In every case y' is contained in M.

Hence one of the cases 1), 2), or 3) always holds.

Section 3. Order

<u>Definition 1.3.1</u>: If $x = y + u$, then $x > y$.

($>$ is greater than.)

<u>Definition 1.3.2</u>: If $y = x + v$, then $x < y$.

($<$ is less than.)

It is readily seen that either $x = y$, or $x > y$, or $x < y$. If $x > y$, then $y < x$, and if $x < y$, then $y > x$.

The symbol $\geq$ means greater than or equal to.

The symbol $\leq$ means less than or equal to.

<u>Theorem 1.3.1</u>: If $x < y$, $y < z$,

then $x < z$.

Proof: Pick a suitable u, v for which

$$y = x + v,$$

$$z = y + u,$$

then $z = (x + v) + u = x + (v + u)$,

then $z > x$,

or $x < z$.

Similarly it follows from $x > y$, $y > z$, that $x > z$, and from $z < y$, $y < x$ that $z < x$.

<u>Theorem 1.3.2</u>: If $x \leq y$, $y < z$, or $x < y$, $y \leq z$

then $x < z$.

Proof: Similar to theorem 1.3.1.

<u>Theorem 1.3.3</u>: If $x \leq y$, $y \leq z$ then $x \leq z$.

Proof: Similar to theorem 1.3.1.

<u>Theorem 1.3.4:</u> $x + y > x.$

Proof: For $(x + y) = x + y = x + u$ where $u = y$, then by definition 1.3.1 $(x + y) > x.$

<u>Theorem 1.3.5:</u> If either $x > y$, or $x = y$, or $x < y$, it follows respectively that either

$x + z > y + z$, or $x + z = y + z$, or $x + z < y + z.$

Proof: 1) Let $x > y$

then $x = y + u,$

$x + z = (y + u) + z = (u + y) + z = u + (y + z) = (y + z) + u$

by definition 1.3.1 $x + z > y + z.$

2) If $x = y$, it follows naturally that

$$x + z = y + z.$$

3) If $x < y$, then $y > x$,

and by part 1) $y + z > x + z,$

then $x + z < y + z.$

<u>Theorem 1.3.6:</u> If either

$x + z > y + z$, or $x + z = y + z$, or $x + z < y + z$,

it follows respectively that either

$$x > y, \text{ or } x = y \text{ or } x < y.$$

Proof: From $x + z > y + z$, we know that either

$$x > y, \text{ or } x = y, \text{ or } x < y.$$

If $x < y$, then $x + z < y + z$, and if $x = z$, $x + z = y + z$, then $x > y$ must hold.

Similar proof for the other cases.

Theorem 1.3.7: If $x \geq y$, $z \geq u$ then $x + z \geq y + u$, where equality holds only if $x = y$, $z = u$.

Proof: From theorem 1.3.5 we have

$$x + z \geq y + z \quad \text{if } x \geq y,$$

then $$x + z \geq y + z = z + y \geq u + y = y + u,$$

then $$x + z \geq y + u.$$

Theorem 1.3.8: $$x \geq 1.$$

Proof: Either $x = 1$ or $x = u'$. If $x = u'$

then $$x = u' = u + 1 > 1.$$

Theorem 1.3.9: If $y > x$, then $y \geq x + 1$.

Proof: $y = x + u$, where $u \geq 1$, then $y \geq x + 1$.

Theorem 1.3.10: If $y < x + 1$, then $y \leq x$.

Proof: Assume $y > x$, then $y \geq x + 1$, but $y < x + 1$, then $y \leq x$.

Theorem 1.3.11: In every set of natural numbers there is a smallest number.

Proof: N is a given class of numbers. M is a class of x such that every number in M $\leq$ every number in N.

$y < y'$, then 1 is contained in M.

Not every x is in M, because for y contained in N, $y + 1 > y$, hence $y + 1$ is not contained M.

There is an m contained in M such that m' is not contained in M. If n is in N, then $m < n$, and $m + 1 \leq n$.

Section 4. Multiplication

Theorem 1.4.1: Every pair of numbers x, y, generate a unique number x y, called the product of x multiplied by y, with the following properties.

1) $x \cdot 1 = x$

2) $x \cdot y' = xy + x$ for all y.

Proof: A) First we will show that for a fixed x, there is at most one possibility xy for all values of y so defined, such that

$x \cdot 1 = x$

$x \cdot y' = xy + x$ for all y.

Let a_y and b_y be defined for all y, such that

$a_1 = x,$ $b_1 = x,$

$a_{y'} = a_y + x,$ $b_{y'} = b_y + x,$

and let M be the class of y for which

$$a_y = b_y.$$

1) $a_1 = x = b_1,$

then 1 is contained in M.

2) For any y contained in M, we have $a_y = b_y$ by definition.

Then $a_{y'} = a_y + x = b_y + x = b_{y'}$,

and y' is contained in M.

M is the class of all natural numbers. The solution is unique, if it exists.

B) We shall now show that xy does exist, as defined by the following properties, for all x and y.

1) $x \cdot 1 = x$

2) $x \cdot y' = xy + x$

Let M be the class of x for which there is exactly one such possibility.

1) For $x = 1$, define $xy = y$.

Then $x \cdot 1 = 1 = x,$

$$xy' = y' = y + 1 = xy + x.$$

1 is contained in M.

2) Now let x be any number in M. Then xy exists by assumption for all y. Define $x'y = xy + y$.

Then $x' 1 = x \cdot 1 + 1 = x + 1 = x'$, and

$x'y' = xy' + y' = (xy + x) + y' = xy + (x + y') = xy + (x+y)'$

$= xy + (x' + y) = xy + (y + x') = (xy + y) + x' = x'y + x'$,

hence x' belongs to M.

Therefore M is the set of all natural numbers, and the proof is complete.

<u>Theorem 1.4.2:</u> $xy = yx$.

Proof: For a given y, let M be the class of x for which $xy = yx$.

1) Since $y \cdot 1 = y$, and $1 \cdot y = y$, by theorem 1.4.1, then

$$y \cdot 1 = 1 \cdot y.$$

1 is contained in M.

2) If M is the class of x for which $xy = yx$,

then $$xy + y = yx + y = yx',$$

and by theorem 1.4.1 $$x'y = xy + y,$$

then $$x'y = y\,x'.$$

Therefore x' is contained in M, and M is the class of all natural numbers.

<u>Theorem 1.4.3:</u> $$x(y + z) = xy + xz.$$

Proof: For a fixed x, y, let M be the class of z for which $$x(y + z) = xy + xz.$$

1) Let $z = 1$,

then $$x(y + 1) = x\,y' = xy + x = xy + x\cdot 1.$$

1 is contained in M.

2) When z is contained in M, then $x(y + z) = xy + xz$.

Then $$x(y + z') = x((y + z)') = x(y + z) + x = (xy + xz) + x$$
$$= xy + (xz + x) = xy + xz'.$$

z' is contained in M.

M contains the class of all natural numbers N.

<u>Theorem 1.4.4:</u> $$(xy)z = x(yz).$$

Proof: Let x and y be fixed, and M be the class of z for which $(xy)z = x(yz)$.

1) If $z = 1$,

then $$(xy)\cdot 1 = xy = x(y\cdot 1).$$

Therefore 1 is contained in M.

2) Let z be contained in M. Then $(xy)z = x(yz)$.

From theorem 1.4.3 $\quad (xy)z' = (xy)z + xy = x(yz) + xy$

$$= x\,(yz + y) = x(yz').$$

Therefore z' is contained in M, and M is the class of all natural numbers N.

<u>Theorem 1.4.5:</u> If $x > y$, or $x = y$, or $x < y$, it follows respectively $xz > yz$, or $xz = yz$, or $xz < yz$.

Proof: 1) For $x > y$, $x = (y + u)$,

then $\quad xz = (\, y + u\,)z = yz + uz > yz$.

Therefore $\qquad xz > yz$.

2) From $x = y$, it follows that $xz = yz$.

3) From $x < y$, follows $y > x$.

From part 1) $\qquad yz > xz$,

then $\qquad xz < yz$.

<u>Theorem 1.4.6:</u> If $xz > yz$, or $xz = yz$, or $xz < yz$, it follows respectively $x > y$, or $x = y$, or $x < y$.

Proof: From $xz > yz$ we know that either $x > y$, or $x = y$, or $x < y$,

but if $x = y$, then by theorem 1.4.5 $xz = yz$,

and if $x < y$, then by theorem 1.4.5 $xz < yz$.

Therefore if $xz > yz$, then $x > y$.

Similar proofs for the other parts.

<u>Theorem 1.4.7:</u> If $x > y$, $z > u$, then $xz > yu$.

Proof: By theorem 1.4.5 $xz > yz$,

and $yz = zy > uy = yu$.

Therefore $xz > yu$.

<u>Theorem 1.4.8:</u> If $x \geq y$, $z > u$, or $x > y$, $z \geq u$,

then $xz > yu$.

Proof: Similar to that of theorem 1.4.7.

<u>Theorem 1.4.9:</u> If $x \geq y$, $z \geq u$, then $xz \geq yu$.

Proof: Similar to that of theorem 1.4.7.

CHAPTER II

FRACTIONS

Section 1. Definitions and Equivalence

Definition 2.1.1: By a fraction $\frac{x_1}{x_2}$ is understood the ordered pair of numbers x_1, x_2.

Definition 2.1.2: $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$ when $x_1 y_2 = y_1 x_2$.

($\sim$ means equivalent.)

Theorem 2.1.1: $\frac{x_1}{x_2} \sim \frac{x_1}{x_2}$.

Proof: $x_1 x_2 = x_1 x_2$.

Theorem 2.1.2: If $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$, then $\frac{y_1}{y_2} \sim \frac{x_1}{x_2}$

Proof: $x_1 y_2 = y_1 x_2$, then $y_1 x_2 = x_1 y_2$.

Theorem 2.1.3: If $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$, $\frac{y_1}{y_2} \sim \frac{z_1}{z_2}$, then $\frac{x_1}{x_2} \sim \frac{z_1}{z_2}$.

Proof: $x_1y_2 = y_1x_2, \quad y_1z_2 = z_1y_2,$

then $$(x_1y_2)(y_1z_2) = (y_1x_2)(z_1y_2).$$

Since
$(xy)(zu) = x(y(zu)) = x((yz)u) = x(u(yz) = (xu)(yz) = (xu)(zy);$

then $$(x_1y_2)(y_1z_2) = (x_1z_2)(y_1y_2),$$

and $$(y_1x_2)(z_1y_2) = (y_1y_2)(z_1x_2)$$

$$= (z_1x_2)(y_1y_2),$$

hence $$(x_1z_2)(y_1y_2) = (z_1x_2)(y_1y_2),$$

and $$x_1z_2 = z_1x_2.$$

<u>Theorem 2.1.4</u>: $\frac{x_1}{x_2} \sim \frac{x_1\,x}{x_2x}.$

Proof: $x_1(x_2x) = x_1(xx_2) = (x_1x)x_2.$

Section 2. Order

Definition 2.2.1: $\frac{x_1}{x_2} > \frac{y_1}{y_2}$, when $x_1y_2 > y_1x_2$.

Definition 2.2.2: $\frac{x_1}{x_2} < \frac{y_1}{y_2}$, when $x_1y_2 < y_1x_2$.

Theorem 2.2.1: There are arbitrary $\frac{x_1}{x_2}$, $\frac{y_1}{y_2}$, so that one of the following cases holds,

$$\frac{x_1}{x_2} \sim \frac{y_1}{y_2}, \text{ or } \frac{x_1}{x_2} > \frac{y_1}{y_2}, \text{ or } \frac{x_1}{x_2} < \frac{y_1}{y_2}.$$

Proof: For any x_1, x_2, y_1, y_2, exactly one of the relations $x_1y_2 =, >, \text{or} <, x_2y_1$ holds.

1) If $\frac{x_1}{x_2}$ $\frac{y_1}{y_2}$, then $x_1y_2 = y_1x_2$.

Similar proofs for the other parts.

Theorem 2.2.2: If $\frac{x_1}{x_2} > \frac{y_1}{y_2}$, then $\frac{y_1}{y_2} < \frac{x_1}{x_2}$.

Proof: Since $x_1y_2 > y_1x_2$, then $y_1x_2 < x_1y_2$.

Theorem 2.2.3: If $\frac{x_1}{x_2} > \frac{y_1}{y_2}$, $\frac{x_1}{x_2} \sim \frac{z_1}{z_2}$, $\frac{y_1}{y_2} \sim \frac{u_1}{u_2}$,

then

$$\frac{z_1}{z_2} > \frac{u_1}{u_2}.$$

Proof: $y_1u_2 = u_1y_2$, $z_1x_2 = x_1z_2$, $x_1y_2 > y_1x_2$,

then $$(y_1u_2)(z_1x_2) = (u_1y_2)(x_1z_2),$$

by theorem 1.4.5

$$(y_1x_2)(z_1u_2) = (u_1z_2)(x_1y_2) > (u_1z_2)(y_1x_2),$$

then by theorem 1.4.6 $z_1u_2 > u_1z_2$.

__Theorem 2.2.5:__ If $\frac{x_1}{x_2} < \frac{y_1}{y_2}$, $\frac{x_1}{x_2} \sim \frac{z_1}{z_2}$, $\frac{y_1}{y_2} \sim \frac{u_1}{u_2}$,

then $$\frac{z_1}{z_2} < \frac{u_1}{u_2}.$$

Proof: Similar to proof of theorem 2.2.4.

__Definition 2.2.3:__ $\frac{x_1}{x_2} \gtrsim \frac{y_1}{y_2}$ means that either

$\frac{x_1}{x_2} > \frac{y_1}{y_2}$, or $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$.

($\gtrsim$ means greater than, or equivalent to.)

__Definition 2.2.4:__ $\frac{x_1}{x_2} \lesssim \frac{y_1}{y_2}$ means that either

$\frac{x_1}{x_2} < \frac{y_1}{y_2}$, or $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$.

($\lesssim$ means less than, or equivalent to.)

<u>Theorem 2.2.6:</u> If $\frac{x_1}{x_2} \gtrsim \frac{y_1}{y_2}$, $\frac{x_1}{x_2} \sim \frac{z_1}{z_2}$, $\frac{y_1}{y_2} \sim \frac{u_1}{u_2}$,

then $$\frac{z_1}{z_2} \gtrsim \frac{u_1}{u_2}.$$

Proof similar to that of theorem 2.2.4.

<u>Theorem 2.2.7:</u> If $\frac{x_1}{x_2} \lesssim \frac{y_1}{y_2}$, $\frac{x_1}{x_2} \sim \frac{z_1}{z_2}$, $\frac{y_1}{y_2} \sim \frac{u_1}{u_2}$,

then $$\frac{z_1}{z_2} \lesssim \frac{u_1}{u_2}.$$

Proof similiar to that of theorem 2.2.4.

<u>Theorem 2.2.8:</u> If $\frac{x_1}{x_2} \gtrsim \frac{y_1}{y_2}$, then $\frac{y_1}{y_2} \lesssim \frac{x_1}{x_2}$.

Proof follows from theorem 2.1.2 and theorem 2.2.1.

<u>Theorem 2.2.9:</u> If $\frac{x_1}{x_2} \lesssim \frac{y_1}{y_2}$, then $\frac{y_1}{y_2} \gtrsim \frac{x_1}{x_2}$.

Proof follows from theorem 2.1.2 and theorem 2.2.2.

Theorem 2.2.10: If $\frac{x_1}{x_2} < \frac{y_1}{y_2}$, $\frac{y_1}{y_2} < \frac{z_1}{z_2}$, then $\frac{x_1}{x_2} < \frac{z_1}{z_2}$.

Proof: $x_1y_2 < y_1x_2$, $y_1z_2 < z_1y_2$,

then $$(x_1y_2)(y_1z_2) < (y_1x_2)(z_1y_2),$$

and $$(x_1z_2)(y_1y_2) < (z_1x_2)(y_1y_2),$$

then $$x_1z_2 < z_1x_2.$$

Theorem 2.2.11: If $\frac{x_1}{x_2} \lesssim \frac{y_1}{y_2}$, $\frac{y_1}{y_2} < \frac{z_1}{z_2}$,

or $\frac{x_1}{x_2} < \frac{y_1}{y_2}$, $\frac{y_1}{y_2} \lesssim \frac{z_1}{z_2}$, then $\frac{x_1}{x_2} < \frac{z_1}{z_2}$.

Proof follows from theorem 2.2.5 and theorem 2.2.10.

Theorem 2.2.12: If $\frac{x_1}{x_2} \lesssim \frac{y_1}{y_2}$, $\frac{y_1}{y_2} \lesssim \frac{z_1}{z_2}$, then $\frac{x_1}{x_2} \lesssim \frac{z_1}{z_2}$.

Proof follows from theorem 2.1.3 and theorem 2.2.11.

Theorem 2.2.13: For each $\frac{x_1}{x_2}$ given there is a

$$\frac{z_1}{z_2} > \frac{x_1}{x_2}.$$

Proof: $(x_1 + x_1)x_2 = x_1x_2 + x_1x_2 > x_1x_2$,

then

$$\frac{x_1 + x_1}{x_2} > \frac{x_1}{x_2}.$$

Theorem 2.2.14: For each $\frac{x_1}{x_2}$ given, there is a $\frac{z_1}{z_2} < \frac{x_1}{x_2}$.

Proof: $x_1x_2 < x_1x_2 + x_1x_2 = x_1(x_2 + x_2)$,

then

$$\frac{x_1}{x_2 + x_2} < \frac{x_1}{x_2}.$$

Theorem 2.2.15: If $\frac{x_1}{x_2} < \frac{y_1}{y_2}$, there is a $\frac{z_1}{z_2}$ such that

$$\frac{x_1}{x_2} < \frac{z_1}{z_2} < \frac{y_1}{y_2}.$$

Proof: $x_1y_2 < y_1x_2$,

then

$x_1x_2 + x_1y_2 < x_1x_2 + y_1x_2$, $x_1y_2 + y_1y_2 < y_1x_2 + y_1y_2$,

$$x_1(x_2 + y_2) < (x_1 + y_1)x_2, \qquad (x_1 + y_1)y_2 < y_1(x_2 + y_2),$$

then
$$\frac{x_1}{x_2} < \frac{x_1 - y_1}{x_2 - y_2} < \frac{y_1}{y_2} \, .$$

Section 3. Addition

Definition 2.3.1: By $\frac{x_1}{x_2} + \frac{y_1}{y_2}$ is understood the fraction

$$\frac{x_1y_2 + y_1x_2}{x_2y_2} .$$

Theorem 2.3.1: If $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$, $\frac{z_1}{z_2} \sim \frac{u_1}{u_2}$, then $\frac{x_1}{x_2} + \frac{z_1}{z_2} \sim \frac{y_1}{y_2} + \frac{u_1}{u_2}$.

Proof: $x_1y_2 = y_1x_2$, $\qquad z_1u_2 = u_1z_2$,

then

$$(x_1y_2)(z_2u_2) = (y_1x_2)(z_2u_2), \quad (z_1u_2)(x_2y_2) = (u_1z_2)(x_2y_2),$$

then

$$(x_1z_2)(y_2u_2) = (y_1u_2)(x_2z_2), \quad (z_1x_2)(y_2u_2) = (u_1y_2)(x_2z_2),$$

$$(x_1z_2)(y_2u_2) + (z_1x_2)(y_2u_2) = (y_1u_2)(x_2z_2) + (u_1y_2)(x_2z_2),$$

$$(x_1z_2 + z_1x_2)(y_2u_2) = (y_1u_2 + u_1y_2)(x_2z_2),$$

$$\frac{x_1z_2 + z_1x_2}{x_2z_2} \quad \frac{y_1u_2 + u_1y_2}{y_2u_2}$$

Theorem 2.3.2: $\frac{x_1}{x} + \frac{x_2}{x} \quad \frac{x_1 + x_2}{x}$.

Proof: By Definition 2.3.1 and theorem 2.1.4,

$$\frac{x_1}{x} + \frac{x_2}{x} \sim \frac{x_1x + x_2x}{xx} \sim \frac{(x_1 + x_2)x}{xx} \sim \frac{x_1 + x_2}{x} .$$

Theorem 2.3.3: $\frac{x_1}{x_2} + \frac{y_1}{y_2} \sim \frac{y_1}{y_2} + \frac{x_1}{x_2}$.

Proof: $\frac{x_1}{x_2} + \frac{y_1}{y_2} \sim \frac{x_1y_2 + y_1x_2}{x_2y_2} \sim \frac{y_1x_2 + x_1y_2}{y_2x_2} \sim \frac{y_1}{y_2} + \frac{x_1}{x_2}$.

Theorem 2.3.4: $\left(\frac{x_1}{x_2} + \frac{y_1}{y_2}\right) + \frac{z_1}{z_2} \sim \frac{x_1}{x_2} + \left(\frac{y_1}{y_2} - \frac{z_1}{z_2}\right)$.

Proof: $\left(\frac{x_1}{x_2} + \frac{y_1}{y_2}\right) + \frac{z_1}{z_2} \sim \frac{x_1y_2 + y_1x_2}{x_2y_2} + \frac{z_1}{z_2}$

$$\frac{(x_1y_2 + y_1x_2)z_2 + z_1(x_2y_2)}{(x_2y_2)z_2} \sim \frac{((x_1y_2)z_2 + (y_1x_2)z_2) + z_1(y_2x_2)}{x_2(y_2z_2)}$$

$$\sim \frac{(x_1(y_2z_2) + (x_2y1)z_2) + (z_1y_2)x_2}{x_2(y_2z_2)}$$

$$\sim \frac{(x_1(y_2z_2) + x_2(y_1z_2)) + (z_1y_2)x_2}{x_2(y_2z_2)}$$

$$\sim \frac{x_1(y_2z_2) + ((y_1z_2)x_2 + (z_1y_2)x_2)}{x_2(y_2z_2)} \sim \frac{x_1(y_2z_2) + (y_1z_2 + z_1y_2)x_2}{x_2(y_2z_2)}$$

$$\frac{x_1}{x_2} + \frac{y_1z_2 - z_1y_2}{y_2z_2} \sim \frac{x_1}{x_2} + \left(\frac{y_1}{y_2} + \frac{z_1}{z_2}\right).$$

Theorem 2.3.5: $\frac{x_1}{x_2} + \frac{y_1}{y_2} > \frac{x_1}{x_2}.$

Proof: $x_1y_2 + y_1x_2 > x_1y_2,$

$$(x_1y_2 + y_1x_2)x_2 > (x_1y_2)x_2 = x_1(y_2x_2) = x_1(x_2y_1),$$

$$\frac{x_1}{x_2} + \frac{y_1}{y_2} \sim \frac{x_1y_2 + y_1x_2}{x_2y_2} > \frac{x_1}{x_2}.$$

Theorem 2.3.6: If $\frac{x_1}{x_2} > \frac{y_1}{y_2}$, then $\frac{x_1}{x_2} + \frac{z_1}{z_2} > \frac{y_1}{y_2} + \frac{z_1}{z_2}$.

Proof: Since $x_1y_2 > y_1x_2$, then $(x_1y_2)z_2 > (y_1x_2)z_2$.

Since $(xy)z = x(yz) = x(zy) = (xz)y$, then $(x_1z_2)y_2 > (y_1x_2)z_2$.

and $(z_1x_2)y_2 = (z_1y_2)x_2$, then $(x_1z_2 - z_1x_2)y_2 > (y_1z_2 - z_1y_2)x_2$,

$$(x_1z_2 + z_1x_2)(y_2z_2) \quad (y_1z_2 + z_1y_2)(x_2z_2),$$

$$\frac{x_1}{x_2} + \frac{z_1}{z_2} \sim \frac{x_1z_2 + z_1x_2}{x_2z_2} > \frac{y_1z_2 + z_1y_2}{y_2z_2} \sim \frac{y_1}{y_2} + \frac{z_1}{z_2}.$$

Theorem 2.3.7: If either $\frac{x_1}{x_2} > \frac{y_1}{y_2}$, or $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$, or $\frac{x_1}{x_2} < \frac{y_1}{y_2}$, it follows respectively that either

$$\frac{x_1}{x_2} + \frac{z_1}{z_2} > \frac{y_1}{y_2} + \frac{z_1}{z_2}, \text{ or } \frac{x_1}{x_2} + \frac{z_1}{z_2} \sim \frac{y_1}{y_2} + \frac{z_1}{z_2}, \text{ or } \frac{x_1}{x_2} + \frac{z_1}{z_2} < \frac{y_1}{y_2} + \frac{z_1}{z2}.$$

Proof: Case 1 follows from theorem 2.3.6.

Case 2 follows from theorem 2.3.1.

Case 3: $\frac{x_1}{x_2} < \frac{y_1}{y_2}$, then $\frac{y_1}{y_2} > \frac{x_1}{x_2}$,

and $\frac{y_1}{y_2} + \frac{z_1}{z_2} > \frac{x_1}{x_2} + \frac{z_1}{z_2}$, then $\frac{x_1}{x_2} + \frac{z_1}{z_2} < \frac{y_1}{y_2} + \frac{z_1}{z_2}$.

Theorem 2.3.8: If $\frac{x_1}{x_2} > \frac{y_1}{y_2}$, $\frac{z_1}{z_2} > \frac{u_1}{u_2}$, then $\frac{x_1}{x_2} + \frac{z_1}{z_2} > \frac{y_1}{y_2} + \frac{u_1}{u_2}$.

Proof: By theorem 2.3.6 $\frac{x_1}{x_2} + \frac{z_1}{z_2} > \frac{y_1}{y_2} + \frac{z_1}{z_2}$,

and

$$\frac{y_1}{y_2} + \frac{z_1}{z_2} \sim \frac{z_1}{z_2} + \frac{y_1}{y_2} > \frac{u_1}{u_2} + \frac{y_1}{y_2} \sim \frac{y_1}{y_2} + \frac{u_1}{u_2},$$

then

$$\frac{x_1}{x_2} + \frac{z_1}{z_2} > \frac{y_1}{y_2} + \frac{u_1}{u_2}.$$

Theorem 2.3.9 If either $\frac{x_1}{x_2} + \frac{z_1}{z_2} > \frac{y_1}{y_2} + \frac{z_1}{z_2}$, or $\frac{x_1}{x_2} + \frac{z_1}{z_2} \sim \frac{y_1}{y_2} + \frac{z_1}{z_2}$

or $\frac{x_1}{x_2} + \frac{z_1}{z_2} < \frac{y_1}{y_2} + \frac{z_1}{z_2}$, then either $\frac{x_1}{x_2} > \frac{y_1}{y_2}$, or $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$,

or $\frac{x_1}{x_2} < \frac{y_1}{y_2}$, respectively.

Proof: We know that for $\frac{x_1}{x_2} + \frac{z_1}{z_2} > \frac{y_1}{y_2} + \frac{z_1}{z_2}$, one of

following must hold either $\frac{x_1}{x_2} > \frac{y_1}{y_2}$, or $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$, or $\frac{x_1}{x_2} < \frac{y_1}{y_2}$.

If $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$, then $\frac{x_1}{x_2} + \frac{z_1}{z_2} \sim \frac{y_1}{y_2} + \frac{z_1}{z_2}$,

or if $\frac{x_1}{x_2} < \frac{y_1}{y_2}$, then $\frac{x_1}{x_2} + \frac{z_1}{z_2} < \frac{y_1}{y_2} + \frac{z_1}{z_2}$, therefore $\frac{x_1}{x_2} > \frac{y_1}{y_2}$.

Similar proofs for the other cases,

<u>Theorem 2.3.10:</u> If $\frac{x_1}{x_2} \gtrsim \frac{y_1}{y_2}$, $\frac{z_1}{z_2} > \frac{u_1}{u_2}$, or $\frac{x_1}{x_2} > \frac{y_1}{y_2}$, $\frac{z_1}{z_2} \gtrsim \frac{u_1}{u_2}$,

then
$$\frac{x_1}{x_2} + \frac{z_1}{z_2} > \frac{y_1}{y_2} + \frac{u_1}{u_2} .$$

Proof follows from theorems 2.3.1, 2.3.6, and 2.3.8.

Theorem 2.3.11: If $\frac{x_1}{x_2} > \frac{y_1}{y_2}$, then $\frac{y_1}{y_2} + \frac{u_1}{u_2} \sim \frac{x_1}{x_2}$ has a

solution $\frac{u_1}{u_2}$. Moreover, if $\frac{u_1}{u_2}$ and $\frac{v_1}{v_2}$ are solutions,

then
$$\frac{u_1}{u_2} \sim \frac{v_1}{v_2}.$$

Proof: By theorem 2.3.9 if $\frac{y_1}{y_2} + \frac{u_1}{u_2} \sim \frac{y_1}{y_2} + \frac{v_1}{v_2}$, then $$\frac{u_1}{u_2} \sim \frac{v_1}{v_2}.$$

Construct a fraction $\frac{u_1}{u_2}$ as follows:

when $x_1y_2 > y_1x_2$, then $x_1y_2 = y_1x_2 + u$, then let $u_1 = u$, and let $u_2 = x_2y_2$. Then $\frac{u_1}{u_2}$ is the solution because

$$\frac{y_1}{y_2} + \frac{u_1}{u_2} \sim \frac{y_1}{y_2} + \frac{u}{x_2y_2} \sim \frac{y_1x_2}{x_2y_2} + \frac{u}{x_2y_2} \sim \frac{y_1x_2 + u}{x_2y_2} \sim \frac{x_1y_2}{x_2y_2} \sim \frac{x_1}{x_2}.$$

Definition 2.3.2. The fraction $\frac{u_1}{u_2}$, constructed in the proof of theorem 2.3.11 is called the difference $\frac{x_1}{x_2}$ minus (-) $\frac{y_1}{y_2}$.

If $$\frac{x_1}{x_2} \sim \frac{y_1}{y_2} + \frac{v_1}{v_2},$$

then $$\frac{v_1}{v_2} \sim \frac{x_1}{x_2} - \frac{y_1}{y_2}.$$

Section 4. Multiplication

Definition 2.4.1: By $\frac{x_1}{x_2}\cdot\frac{y_1}{y_2}$ is understood the fraction $\frac{x_1y_1}{x_2y_2}$. This fraction is called the product of $\frac{x_1}{x_2}$ by $\frac{y_1}{y_2}$.

Theorem 2.4.1: If $\frac{x_1}{x_2}\sim\frac{y_1}{y_2}$, $\frac{z_1}{z_2}\sim\frac{u_1}{u_2}$, then $\frac{x_1}{x_2}\,\frac{z_1}{z_2}\sim\frac{y_1}{y_2}\,\frac{u_1}{u_2}$.

Proof: $x_1y_2 = y_1x_2$, $z_1u_2 = u_1z_2$, then $(x_1y_2)(z_1u_2) = (y_1x_2)(u_1z_2)$, $(x_1z_1)(y_2u_2) = (y_1u_1)(x_2z_2)$, and

$$\frac{x_1}{x_2}\,\frac{z_1}{z_2}\sim\frac{y_1u_1}{y_2u_2}.$$

Theorem 2.4.2:

$$\frac{x_1}{x_2}\,\frac{y_1}{y_2}\sim\frac{y_1}{y_2}\,\frac{x_1}{x_2}.$$

Proof: $$\frac{x_1}{x_2}\,\frac{y_1}{y_2}\sim\frac{x_1y_1}{x_2y_2}\sim\frac{y_1x_1}{y_2x_2}\sim\frac{y_1}{y_2}\,\frac{x_1}{x_2}.$$

Theorem 2.4.3:

$$\left(\frac{x_1}{x_2}\,\frac{y_1}{y_2}\right)\frac{z_1}{z_2}\sim\frac{x_1}{x_2}\left(\frac{y_1}{y_2}\,\frac{z_1}{z_2}\right).$$

Proof: $$\left(\frac{x_1}{x_2}\,\frac{y_1}{y_2}\right)\frac{z_1}{z_2}\sim\frac{x_1y_1}{x_2y_2}\,\frac{z_1}{z_2}\sim\frac{(x_1y_1)z_1}{(x_2y_2)z_2}\sim\frac{x_1(y_1z_1)}{x_2(y_2z_2)}$$

$$\sim\frac{x_1}{x_2}\,\frac{y_1z_1}{y_2z_2}\sim\frac{x_1}{x_2}\left(\frac{y_1}{y_2}\,\frac{z_1}{z_2}\right).$$

Theorem 2.4.4:

$$\frac{x_1}{x_2}\left(\frac{y_1}{y_2}+\frac{z_1}{z_2}\right)\sim\frac{x_1}{x_2}\,\frac{y_1}{y_2}+\frac{x_1}{x_2}\,\frac{z_1}{z_2}.$$

Proof: $$\frac{x_1}{x_2}\left(\frac{y_1}{y_2}+\frac{z_1}{z_2}\right)\sim\frac{x_1}{x_2}\,\frac{y_1z_2+z_1y_2}{y_2z_2}\sim\frac{x_1(y_1z_2+z_1y_2)}{x_2(y_2z_2)}$$

$$\frac{x_1(y_1z_2)+x_1(z_1y_2)}{x_2(y_2z_2)} \sim \frac{x_1(y_1z_2)}{x_2(y_2z_2)} + \frac{x_1(z_1y_2)}{x_2(y_2z_2)}$$

$$\sim \frac{(x_1y_1)z_2}{(x_2y_2)z_2} + \frac{(x_1z_1)y_2}{(x_2z_2)y_2} \sim \frac{x_1y_1}{x_2y_2} + \frac{x_1z_1}{x_2z_2} \sim \frac{x_1}{x_2}\frac{y_1}{y_2} + \frac{x_1}{x_2}\frac{z_1}{z_2}.$$

Theorem 2.4.5: If $\frac{x_1}{x_2} > \frac{y_1}{y_2}$, or $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$, or $\frac{x_1}{x_2} < \frac{y_1}{y_2}$,

then it follows respectively that either

$$\frac{x_1}{x_2}\frac{z_1}{z_2} > \frac{y_1}{y_2}\frac{z_1}{z_2}, \quad \text{or} \quad \frac{x_1}{x_2}\frac{z_1}{z_2} \sim \frac{y_1}{y_2}\frac{z_1}{z_2}, \quad \text{or} \quad \frac{x_1}{x_2}\frac{z_1}{z_2} < \frac{y_1}{y_2}\frac{z_1}{z_2}.$$

Proof: 1) If $\frac{x_1}{x_2} > \frac{y_1}{y_2}$ then $x_1y_2 > y_1x_2$,

$$(x_1y_2)(z_1z_2) > (y_1x_2)(z_1z_2), \quad (x_1z_1)(y_2z_2) > (y_1z_1)(x_2z_2)$$

$$\frac{x_1}{x_2}\frac{z_1}{z_2} \sim \frac{x_1z_1}{x_2z_2} > \frac{y_1z_1}{y_2z_2} \sim \frac{y_1}{y_2}\frac{z_1}{z_2}.$$

2) If $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$, then $\frac{x_1}{x_2}\frac{z_1}{z_2} \sim \frac{y_1}{y_2}\frac{z_1}{z_2}$ by theorem 2.4.1.

3) If $\frac{x_1}{x_2} < \frac{y_1}{y_2}$, then $\frac{y_1}{y_2} > \frac{x_1}{x_2}$, and by case 1)

$\frac{y_1}{y_2}\frac{z_1}{z_2} > \frac{x_1}{x_2}\frac{z_1}{z_2}$, then $\frac{x_1}{x_2}\frac{z_1}{z_2} < \frac{y_1}{y_2}\frac{z_1}{z_2}$.

<u>Theorem 2.4.6</u>: If $\frac{x_1}{x_2}\frac{z_1}{z_2} > \frac{y_1}{y_2}\frac{z_1}{z_2}$, or $\frac{x_1}{x_2}\frac{z_1}{z_2} \sim \frac{y_1}{y_2}\frac{z_1}{z_2}$, or

$\frac{x_1}{x_2}\frac{z_1}{z_2} < \frac{y_1}{y_2}\frac{z_1}{z_2}$, then it follows respectively $\frac{x_1}{x_2} > \frac{y_1}{y_2}$,

or $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$, or $\frac{x_1}{x_2} < \frac{y_1}{y_2}$.

Proof follows from theorem 2.4.5, in as much as the cases are mutually exclusive, all possibilities are exhausted.

<u>Theorem 2.4.7</u>: If $\frac{x_1}{x_2} > \frac{y_1}{y_2}$, $\frac{z_1}{z_2} > \frac{u_1}{u_2}$, then $\frac{x_1}{x_2}\,\frac{z_1}{z_2} > \frac{y_1}{y_2}\,\frac{u_1}{u_2}$.

Proof: By theorem 2.4.5 $\frac{x_1}{x_2}\,\frac{z_1}{z_2} > \frac{y_1}{y_2}\,\frac{z_1}{z_2}$, and

$\frac{y_1}{y_2}\,\frac{z_1}{z_2} \sim \frac{z_1}{z_2}\,\frac{y_1}{y_2} > \frac{u_1}{u_2}\,\frac{y_1}{y_2} \sim \frac{y_1}{y_2}\,\frac{u_1}{u_2}$, then $\frac{x_1}{x_2}\,\frac{z_1}{z_2} > \frac{y_1}{y_2}\,\frac{u_1}{u_2}$.

<u>Theorem 2.4.8</u>: If $\frac{x_1}{x_2} \gtrsim \frac{y_1}{y_2}$, $\frac{z_1}{z_2} > \frac{u_1}{u_2}$, or $\frac{x_1}{x_2} > \frac{y_1}{y_2}$, $\frac{z_1}{z_2} \gtrsim \frac{u_1}{u_2}$,

then $\frac{x_1}{x_2}\,\frac{z_1}{z_2} > \frac{y_1}{y_2}\,\frac{u_1}{u_2}$.

Proof follows from theorems 2.4.1, 2.4.5, and 2.4.7.

<u>Theorem 2.4.9</u>: If $\frac{x_1}{x_2} \gtrsim \frac{y_1}{y_2}$, $\frac{z_1}{z_2} \gtrsim \frac{u_1}{u_2}$, then $\frac{x_1}{x_2}\,\frac{z_1}{z_2} \gtrsim \frac{y_1}{y_2}\,\frac{u_1}{u_2}$.

Proof follows from theorems 2.4.1 and 2.4.8.

<u>Theorem 2.4.10</u>: Given any two fractions $\frac{x_1}{x_2}$ and $\frac{y_1}{y_2}$, there exists a fraction $\frac{u_1}{u_2}$ such that $\frac{y_1}{y_2}\,\frac{u_1}{u_2} \sim \frac{x_1}{x_2}$. If also $\frac{y_1}{y_2}\,\frac{v_1}{v_2} \sim \frac{x_1}{x_2}$, then $\frac{u_1}{u_2} \sim \frac{v_1}{v_2}$.

Proof: $\frac{u_1}{u_2}$ is unique by theorem 2.4.6. Define $u_1 = x_1y_2$, and $u_2 = x_2y_1$. Then $\frac{u_1}{u_2}\,\frac{y_1}{y_2} \sim \frac{x_1y_2}{x_2y_1}\,\frac{y_1}{y_2} \sim \frac{x_1(y_1y_2)}{x_2(y_1y_2)} \sim \frac{x_1}{x_2}$.

Section 5. Rational Numbers and Whole Numbers

Definition 2.5.1: A rational number is the class of all fractions equivalent to a given fraction.

U, V, W, X, Y, and Z will be used to denote rational numbers.

Definition 2.5.2: $X = Y$ when both classes contain the same fraction. Otherwise, $X \neq Y$.

In view of the definition of a rational number, and the properties of fractions previously established, the following theorems are evident.

Theorem 2.5.1: $X = X$.

Theorem 2.5.2: If $X = Y$, then $Y = X$.

Theorem 2.5.3: If $X = Y$, $Y = Z$, then $X = Z$.

Definition 2.5.3: $X > Y$ if fractions $\frac{x_1}{x_2}$ and $\frac{y_1}{y_2}$ can be chosen from X and Y respectively such that $\frac{x_1}{x_2} > \frac{y_1}{y_2}$.

Definition 2.5.4: $X < Y$ if fractions $\frac{x_1}{x_2}$ and $\frac{y_1}{y_2}$ can be chosen from X and Y respectively such that $\frac{x_1}{x_2} < \frac{y_1}{y_2}$.

Theorem 2.5.4: If X and Y are arbitrary, then only one of the following cases holds. $X = Y$, or $X > Y$, or $X < Y$.

Proof follows from theorem 2.2.1.

Theorem 2.5.5: If $X > Y$, then $Y < X$.

Proof: Theorem 2.2.2.

<u>Theorem 2.5.6:</u> If $X < Y$, then $Y > X$.

Proof: Theorem 2.2.3.

<u>Definition 2.5.5:</u> $X \geq Y$ means either $X > Y$, or $X = Y$.

<u>Definition 2.5.6:</u> $X \leq Y$ means either $X < Y$, or $X = Y$.

<u>Theorem 2.5.7:</u> If $X \geq Y$, then $Y \leq X$.

Proof: Theorem 2.2.8.

<u>Theorem 2.5.8:</u> If $X \leq Y$, then $Y \geq X$.

Proof: Theorem 2.2.9.

<u>Theorem 2.5.9:</u> If $X < Y$, $Y < Z$, then $X < Z$.

Proof: Theorem 2.2.10.

<u>Theorem 2.5.10:</u> If $X \leq Y$, $Y < Z$, or $X < Y$, $Y \leq Z$, then $X < Z$.

Proof: Theorem 2.2.11.

<u>Theorem 2.5.11:</u> If $X \leq Y$, $Y \leq Z$ then $X \leq Z$.

Proof: Theorem 2.2.12.

<u>Theorem 2.5.12:</u> For any given X there is a $Z > X$.

Proof: Theorem 2.2.13.

<u>Theorem 2.5.13:</u> For any given X there is a $Z < X$. Thus, there is no smallest rational number.

Proof: Theorem 2.2.14.

<u>Theorem 2.5.14:</u> If $X < Y$, then there is a Z such that

$$X < Z < Y.$$

Proof: Theorem 2.2.15.

Definition 2.5.7: $X + Y$ means the class defined by the sum obtained by adding one of the fractions of Y to one of the fractions of X.

Theorem 2.5.15: $X + Y = Y + X$.

Proof: Theorem 2.3.3.

Theorem 2.5.16: $(X + Y) + Z = X + (Y + Z)$.

Proof: Theorem 2.3.4.

Theorem 2.5.17: $X + Y > X$

Proof: Theorem 2.3.5.

Theorem 2.5.18: If $X > Y$, then $X + Z > Y + Z$.

Proof: Theorem 2.3.6.

Theorem 2.5.19: If $X > Y$, or $X = Y$, or $X < Y$, then it follows respectively that $X + Z > Y + Z$, or $X + Z = Y + Z$, or $X + Z < Y + Z$.

P Proof: Theorem 2.3.7.

Theorem 2.5.20: If $X + Z > Y + Z$, or $X + Z = Y + Z$, or $X + Z < Y + Z$, then it follows respectively that $X > Y$, or $X = Y$, or $X < Y$.

Proof: Theorem 2.3.9.

Theorem 2.5.21: If $X > Y$, $Z > U$, then $X + Z > Y + U$.

Proof: Theorem 2.3.8.

Theorem 2.5.22. If $X \geq Y$, $Z > U$, or $X > Y$, $Z \geq U$ then $X + Z > Y + U$.

Proof: Theorem 2.3.10:

<u>Theorem 2.5.23</u>: If $X > Y$, there is only one U such that $Y \vdash U = X$, and $X > U$.

Proof: Theorem 2.3.11.

<u>Definition 2.5.8</u>: The U of theorem 2.5.23 is the difference obtained by subtracting Y from X, and is written $U = X \dashv Y$.

<u>Definition 2.5.9</u>: XY means the class defined by the product obtained from multiplying one of the fractions of X by one of the fractions of Y.

<u>Theorem 2.5.24</u>: $XY = YX$.

Proof: Theorem 2.4.2.

<u>Theorem 2.5.25</u>: $(XY)Z = X(YZ)$.

Proof: Theorem 2.4.3.

<u>Theorem 2.5.26</u>: $X(Y + Z) = XY + XZ$.

Proof: Theorem 2.4.4.

<u>Theorem 2.5.27</u>: If $X > Y$, or $X = Y$, or $X < Y$, then it follows respectively that $XZ > YZ$, or $XZ = YZ$, or $XZ < YZ$.

Proof: Theorem 2.4.5.

<u>Theorem 2.5.28</u>: If $XZ > YZ$, or $XZ = YZ$, or $XZ < YZ$, then it follows respectively that $X > Y$, or $X = Y$, or $X < Y$.

Proof: Theorem 2.4.6.

<u>Theorem 2.5.29</u>: If $X > Y$, $Z > U$, then $XZ > YU$.

Proof: Theorem 2.4.7.

<u>Theorem 2.5.30</u>: If $X \geq Y$, $Z > U$, or $X > Y$, $Z \geq U$, then

$$XZ > YU.$$

Proof: Theorem 2.4.8.

<u>Theorem 2.5.31</u>: If $X \geq Y$, $Z \geq U$, then $XZ \geq YU$.

Proof: Theorem 2.4.9.

<u>Theorem 2.5.32</u>: The equation $YU = X$, where X and Y are given, has only one solution, U.

Proof: Theorem 2.4.10.

<u>Theorem 2.5.33</u>: If $\frac{x}{1} > \frac{y}{1}$, or $\frac{x}{1} \sim \frac{y}{1}$, or $\frac{x}{1} < \frac{y}{1}$, then it follows respectively that either $x > y$, or $x = y$, or

$$x < y.$$

Proof: If $\frac{x}{1} > \frac{y}{1}$, then $x = x \cdot 1 > y \cdot 1 = y$; therefore $x > y$.

If $\frac{x}{1} \sim \frac{y}{1}$, then $x = x \cdot 1 = y \cdot 1 = y$, and $x = y$.

If $\frac{x}{1} < \frac{y}{1}$, then $x = x \cdot 1 \quad y \cdot 1 = y$, and $x < y$.

<u>Definition 2.5.10</u>: A rational number is called a whole number when there is a fraction of the form $\frac{x}{1}$ contained in the class.

<u>Theorem 2.5.34</u>: $\frac{x}{1} + \frac{y}{1} \sim \frac{x + y}{1}$, $\frac{x}{1} \frac{y}{1} \sim \frac{xy}{1}$.

Proof: 1 by theorem 2.3.2. 2 by definition 2.4.1.

<u>Theorem 2.5.35</u>: The class of all whole numbers satisfies the postulates 1 through 5 of natural numbers, if instead of 1 we write the class $\frac{1}{1}$, and instead of x we write

the class $\frac{x}{1}$, and instead of x' we write the class $\frac{x'}{1}$.

Proof: Let W be the class of all whole numbers.

1. The class $\frac{1}{1}$ is a whole number.

2. For each whole number contained in W, there is a consequent whole number contained in W.

If $\frac{x}{1} \sim \frac{y}{1}$, then $\frac{x'}{1} \sim \frac{y'}{1}$.

3. The class $\frac{1}{1}$ is not the consequence of any number.

4. If the class of $\frac{x'}{1} \sim \frac{y'}{1}$, then x' = y', and x = y, then

$$\frac{x}{1} \sim \frac{y}{1}.$$

5. Let N be a class of whole numbers withe the properties:

a. The class $\frac{1}{1}$ belongs to N.

b. If the class $\frac{x}{1}$ belongs to N, then $\frac{x'}{1}$ belongs to N.

Now let N be the class of x for which the class $\frac{x}{1}$ belongs to N. Then 1 is in N, and if x is in N, then x' is in N. Thus every natural number belongs to N, and every whole number belongs to N.

<u>Definition 2.5.11:</u> If x refers to a whole number, the the whole class $\frac{x}{1}$ is given.

<u>Theorem 2.5.36:</u> If Z is the fraction $\frac{x}{y}$, a rational number, then $yZ = x$.

Proof: $\frac{y}{1}\frac{x}{y} \sim \frac{xy}{1y} \sim \frac{x}{1} \sim x.$

<u>Definition 2.5.12:</u> The U of theorem 2.5.32 is the quotient

of X divided by Y.

Theorem 2.5.37: If X and Y are given, there is a z such that

$$zX > Y.$$

Proof: $\frac{Y}{X}$ is a rational number; from theorem 2.5.12 it follows that the whole numbers z, v,

$$\frac{z}{v} > \frac{Y}{X}$$

where $v \geq 1$ by theorem 2.5.33, then by theorem 2.5.27

$$zX = Xz = X(\frac{z}{v}\ v) = (X\frac{z}{v})v \geq (X\frac{z}{v})\ 1 = X\frac{z}{v}\quad X\frac{Y}{X} = Y.$$

CHAPTER 3

THE CUT

Section 1. Definition

Definition 3.1.1: A set A of rational numbers is called a cut when

a. There is a rational number in A, but not every rational number is in A.

b. Every rational number in A is smaller than every rational number in B.

c. There is no largest rational number in A.

We will use a, b, c, d, e, f, and g to denote cuts.

The set A is called the lower class and the numbers in A are called lower numbers. The numbers not in A are called upper numbers and make up what is called the upper class.

The following theorems concerning cuts are evident in view of the definitions of rational numbers and cuts.

Theorem 3.1.1: $a = a$.

Theorem 3.1.2: If $a = b$, then $b = a$.

<u>Theorem 3.1.3:</u> If $a = b$, and $b = c$, then $a = c$.

<u>Theorem 3.1.4:</u> If X is an upper number of a, and $X_1 > X$, then X_1 is also an upper number of a.

Proof follows from part b of definition 3.1.1.

<u>Theorem 3.1.5:</u> If X is a lower number of a, and $X_1 < X$, then X_1 is also a lower number of a.

Proof follows from part b of definition 3.1.1.

Section 2. Order

Definition 3.2.1: Let a and b be cuts. Then $a > b$ if there is a lower number of a which is an upper number of b.

Definition 3.2.2: Let a and b be cuts. Then $a < b$ if there is an upper number of a which is a lower number of b.

The following theorems concerning cuts are now readily established.

Theorem 3.2.1: If $a > b$, then $b < a$.

Theorem 3.2.2: If $a < b$, then $b > a$.

Theorem 3.2.3: Let a and b be arbitrary, then they must fall into one and only one of the following classes,

$a = b$, $a > b$, $a < b$.

Definition 3.2.3: $a \geq b$ means either $a > b$, or $a = b$.

Definition 3.2.4: $a \leq b$ means either $a < b$, or $a = b$.

Theorem 3.2.4: If $a \geq b$, then $b \leq a$.

Theorem 3.2.5: If $a \leq b$, then $b \geq a$.

Theorem 3.2.6: If $a < b$, $b < c$, then $a < c$.

Theorem 3.2.7: If $a \leq b$, $b < c$, or $a < b$, $b \leq c$, then $a < c$.

Theorem 3.2.8: If $a \leq b$, $b \leq c$, then $a \leq c$.

Section 3. Addition

<u>Theorem 3.3.1:</u> Let a and b be cuts. Then (I) the set of all numbers representable in the form $X + Y$, where X is a lower number of a and Y is a lower number of b, is a cut.

(II) No number of this set may be represented as the sum of an upper number of a and an upper number of b.

Proof: Let X be a lower number of a and Y be a lower number of b. Then $X + Y$ is in the set.

Given some X_1 which is an upper number of a, and some Y_1 which is an upper number of b for all corresponding lower numbers X, Y contained in a and b respectively. Since $X < X_1$, $Y < Y_1$, then $X + Y \quad X_1 + Y_1$. Therefore $X_1 + Y_1 \neq X + Y$, and $X_1 + Y_1$ is not contained in the set, proving II. Thus the set satisfies condition a of definition 3.1.1.

To show that condition b is satisfied, let Z be a number such that $Z < X + Y$.

Then $(X + Y)\frac{Z}{X + Y} < (X + Y)\cdot 1$, by theorem 2.5.28 $\frac{Z}{X + Y} < 1$, then by theorem 2.5.27

$$X\frac{Z}{X + Y} < X\cdot 1 = X, \quad \text{and} \quad Y\frac{Z}{X + Y} < Y\cdot 1 = Y.$$

Thus $X\frac{Z}{X + Y}$ and $Y\frac{Z}{X + Y}$ are lower numbers of a and b respectively.

But the sum of these two rational numbers is Z, hence Z is in the set.

Let X_1 be a lower number of a such that $X_1 > X$. Then $X_1 + Y$ is in the set and the set is not void. Hence the set, $X + Y$, is a cut.

<u>Definition 3.3.1</u>: According to theorem 3.3.1 it is possible to construct a cut of the form $a + b$. It is called the sum obtained by the addition of b to a resulting in a cut.

We then have the following theorems.

<u>Theorem 3.3.2</u>: $a + b = b + a$.

<u>Theorem 3.3.3</u>: $(a + b) + c = a + (b + c)$.

<u>Theorem 3.3.4</u>: In every given cut, for any rational number A, there is a lower number X and an upper number U such that $U - X = A$.

Proof: Let X_1 be a lower number, and consider all rational numbers of the form $X_1 + nA$, where n is a whole number.

Let Y be an upper number, then $Y > X_1$.

By theorem 2.5.37, there is an n such that

$$nA > Y + X_1,$$

and $X_1 + nA > (Y - X_1) + X_1 = Y$,

therefore $X_1 + nA$ is not a lower number for all n.

In the set of n, for which $X_1 + nA$ is an upper number, there is some smallest whole number, u, by theorem 2.5.37. If $u = 1$, let $X = X_1$, then $U = X_1 + A$, whence $A = U - X$.

If $u > 1$, let $X = X_1 + (u - 1)A$, then

$U = X_1 + uA = X + A$.

Now X is a lower number and U is an upper number, hence

$$U - X = A.$$

<u>Theorem 3.3.5:</u> $a + b > a$.

<u>Theorem 3.3.6:</u> If $a > b$, then $a + c > b + c$.

<u>Theorem 3.3.7:</u> If $a > b$, or $a = b$, or $a < b$, then it follows respectively that $a + c > b + c$, or $a + c = b + c$, or

$$a + c < b + c.$$

<u>Theorem 3.3.8:</u> If $a + c > b + c$, or $a + c = b + c$, or $a + c < b + c$, then it follows respectively that

$a > b$, or $a = b$, or $a < b$.

<u>Theorem 3.3.9:</u> If $a > b$, $c > d$, then $a + c > b + d$.

<u>Theorem 3.3.10:</u> If $a \geq b$, $c > d$, or $a > b$, $c \geq d$, then

$$a + c > b + d.$$

<u>Theorem 3.3.11:</u> If $a \geq b$, $c \geq d$, then $a + c \geq b + d$.

<u>Theorem 3.3.12:</u> If $a > b$, then there is some c such that

$$b + c = a.$$

Proof: I) There is at most one solution, for if

$c_1 \neq c_2$, then $b + c_1 \neq b + c_2$.

II) First it must be shown that the set of rational numbers of the form $X - Y$, where X is a lower number of a and Y is an upper number of b, $X > Y$, constitutes a cut.

1) We know by theorem 2.5.23 for $X > Y$,

$$X - Y \text{ exists.}$$

No upper number X_1 of a can have the form $X - Y$ because

$$X - Y < (X - Y) + Y = X < X_1,$$

hence not all numbers are in the set $(X - Y)$.

2) If $X - Y$ is defined in the above manner, then if for some rational number U, $U < X - Y$, then

$$U + Y < (X - Y) + Y = X.$$

Let $U + Y = X_2$, where X_2 is a lower number of a.

Then $U = X_2 - Y$, and is contained in the set $(X - Y)$, hence for any $X - Y$, we can find a smaller number in the set.

3) Since a is a cut, it is possible to choose from a a lower number X_3 such that $X_3 > X$.

Then $(X_3 - Y) + Y > (X - Y) + Y$,

and $X_3 - Y > X - Y$.

Therefore $X_3 - Y$ is a larger number in the set then $X - Y$.

Thus by 1, 2, and 3, $X - Y$ is a cut. Let it be called c.

Now we have to prove that $b + c = a$. To do so the following conditions must be satisfied.

a) Every lower number of $b + c$ must be a lower number of a.

b) Every lower number of a must be a lower number of b + c.

Proof of a: Every lower number of b + c is of the form $(X - Y) + Y_1$, where X is a lower number of a, Y is an upper number of b, and Y_1 is a lower number of b, and $X > Y$.

Now $Y > Y_1$,

$((X - Y) + Y_1) + (Y - Y_1) = (X - Y) + (Y_1 + (Y - Y_1))$

$= (X - Y) + Y = X, \quad (X - Y) + Y_1 < X,$

then $(X - Y) + Y_1$ is a lower number of a.

Proof of b: 1) Each Y considered above is also a lower number of a. Then, if X is a lower number of a such that $X > Y$, then by theorem 3.3.4 there is a lower number Y_1 of b, and an upper number Y_2 of b such that $Y_2 - Y_1 = X - Y$. Then since $Y > Y_1$

$Y_2 + (Y - Y_1) = ((X - Y) + Y_1) + (Y - Y_1)$

$= (X - Y) + Y = X, \quad (X - Y) + Y_1 < X,$

$Y_2 - Y_1 = X - Y$, and $Y - Y_1 = X - Y_2$, then

$$Y = (Y - Y_1) + Y_1 = (X - Y_2) + Y_1,$$

then Y is a lower number of b + c.

2) Each lower number of a which is also a

lower number of b is less than each lower number of a considered in part 1, and hence is also in b + c.

This completes the proof.

Definition 3.3.2: The c of theorem 3.3.12 is the difference obtained by subtracting b from a, and is written

$$c = b - a.$$

Section 4. Multiplication

Theorem 3.4.1: I) Let a and b be cuts. Then the set of rational numbers of the form XY, where X is a lower number of a and Y is a lower number of b, form a cut.

II) No number in the set can be represented by the product of an upper number of a and an upper number of b.

Proof: 1. Let X be any lower number of a and Y be any lower number of b. Then XY is in the set.

Pick an X_1 which is an upper number of a, and a Y_1 which is an upper number of b. Since $X < X_1$, $Y < Y_1$, then $XY < X_1Y_1$, and $X_1Y_1 \neq XY$. Thus X_1Y_1 is not contained in the set, and the set does not contain all rational numbers.

2. If X is a lower number of a and Y is a lower number of b, and $Z < XY$. Then $X(\frac{1}{X}Z) = (X\frac{1}{X})\, Z = 1 \cdot Z = Z$,

$$\frac{Z}{X} = \frac{1}{X}Z < \frac{1}{X}(XY) = (\frac{1}{X}X)Y = Y.$$

Thus $\frac{Z}{X}$ is a lower number of b, and since $Z = X\frac{Z}{X}$, Z is contained in the set XY.

3. For any given XY in the set, choose a lower number X_1 of a such that $X_1 > X$. Then $X_1Y > XY$, hence there is a larger number than XY in the set.

Definition 3.4.1: The cut constructed in theorem 3.4.1 is called ab. It is the product of a times b and the multiplication results in a unique cut.

In view of the above definition and the properties of rational numbers the following theorems are evident.

<u>Theorem 3.4.2:</u> $ab = ba$.

<u>Theorem 3.4.3:</u> $(ab)c = a(bc)$.

<u>Theorem 3.4.4:</u> $a(b + c) = ab + ac$.

<u>Theorem 3.4.5:</u> If $a > b$, or $a = b$, or $a < b$, then it follows respectively that either

$ac > bc$ or $ac = bc$, or $ac < bc$.

<u>Theorem 3.4.6:</u> If $ac > bc$, or $ac = bc$, or $ac < bc$, then it follows respectively that $a > b$, or $a = b$, or $a < b$.

<u>Theorem 3.4.7:</u> If $a > b$, $c > d$, then $ac > bd$.

<u>Theorem 3.4.8:</u> If $a \geq b$, $c > d$, or $a > b$, $c \geq d$, then $ac > bd$.

<u>Theorem 3.4.9:</u> If $a \geq b$, $c \geq d$, then $ac \geq bd$.

<u>Theorem 3.4.10:</u> For every rational number R, the set of rational numbers less than R forms a cut.

Proof: By theorem 2.5.13 there is an $X < R$. R itself is not less than R.

If $X < R$, $X_1 \geq R$, then $X < X_1$.

If $X < R$, then by theorem 2.5.14, there is an X_1 such that $X < X_1 < R$.

<u>Definition 3.4.2:</u> The cut constructed in theorem 3.4.10 will be designated by R^*.

<u>Theorem 3.4.11:</u> $a \cdot 1^* = a$.

Proof: $a \cdot 1^*$ is the set of XY, where X is a lower number of a and $Y < 1$.

Every such number of the form XY is less than X, and thus is a lower number of a.

Conversely, for any lower number of a, say X, it is possible to choose from a a lower number $X_1 > X$. Put $Y = \frac{1}{X_1} \cdot X$, then $Y < \frac{1}{X_1} X_1 = 1$, and $X = X_1 Y$.

Hence every lower number of a is a lower number of $a \cdot 1^*$.

<u>Theorem 3.4.12</u>: If a is given then there is a unique b such that $ab = 1^*$.

Proof: Let us consider the set of all numbers of the form $\frac{1}{X}$, where X is any upper number of a with the possible exception of the smallest one(if there is such a smallest). We will show that the class is a cut.

1) There is a number in this set. For when X is an upper number of a, X + X is also an upper number, but not the smallest. Then $\frac{1}{X + X}$ is contained in the set. Therefore the set is not void.

There is a rational number not contained in the set. For, if X_1 is a lower number of a, then for all upper numbers X contained in a $X \neq X_1$. Then

$$X \frac{1}{X} = 1 = X_1 \frac{1}{X_1}. \qquad \frac{1}{X} \neq \frac{1}{X_1}.$$

Hence $\frac{1}{X_1}$ is not contained in the set. The set does not contain all numbers.

2) Let $\frac{1}{X}$ be a number in the set, where X is an upper number of a, and suppose that $U < \frac{1}{X}$.
Then $UX < \frac{1}{X}X = 1 = U\frac{1}{U}$, whence $X < \frac{1}{U}$. Then $\frac{1}{U}$ is an upper number of a and is not the smallest.
Then since $U\frac{1}{U} = 1$, $U = \frac{1}{\frac{1}{U}}$, and U is also contained in the set.

3) If X is an upper number of a, (but not the smallest), there is another upper number X_1 of a such that $X_1 < X$ and by theorem 2.5.14, there is an X_2 such that

$$X_1 < X_2 < X.$$

Then X_2 is an upper number of a and is not the smallest, and from $X_2\frac{1}{X} < X\frac{1}{X} = 1 = X_2\frac{1}{X_2}$ it follows that $\frac{1}{X_2} > \frac{1}{X}$.

For each number in the set, there is a number which is larger. By definition 3.1.1 the set is a cut. Let it be called b.

Now we must prove that $ab = 1^*$. To do so we must show:

A) Any lower number contained in $ab < 1$.

B) Any rational number less than 1 is a lower number of ab.

Proof of A: Any lower number of ab may be written in the form $X\frac{1}{X_1}$, where X is a lower number of a, X_1 is an upper number of a. Since $X < X_1$, then

$$X\frac{1}{X_1} < X_1\frac{1}{X_1} = 1.$$ Then every lower number of ab < 1.

Proof of B: Let $U < 1$. For any lower number X of a, by theorem 3.3.4 there is a lower number X_1 contained in a, and an upper number X_2 of a such that $X_2 - X_1 = (1 - U)X$. Then since $X_2 > X$, $X_2 - X_1 < (1 - U)X_2$.

Then $(X_2 - X_1) + UX_2 < (1 - U)\ X_2 + UX_2 = X_2 = (X_2 - X_1) + X_1$,

$$UX_2 < X_1, \qquad X_2 = (\tfrac{1}{U}U)X_2 = \tfrac{1}{U}(UX_2) < \tfrac{1}{U}X_1 = \frac{X_1}{U}.$$

Therefore $\frac{X_1}{U}$ is also an upper number of a, but is not the smallest. From $U\frac{X_1}{U} = X_1$, it follows that

$$U = \frac{X_1}{\frac{X_1}{U}} = X_1\frac{1}{\frac{X_1}{U}};$$

Then since X_1 is a lower number of a, and $\frac{1}{\frac{X_1}{U}}$ is a lower number of b, then U is a lower number of ab. This completes the proof.

<u>Theorem 3.4.13</u>: In the equation $bc = a$, where a and b are given, there is a unique solution c.

Proof: 1) There is at most one solution, because if $c_1 = c_2$, then $bc_1 = bc_2$.

2) If d is found by theorem 3.4.12 such that $bd = 1^*$, then let $c = da$. Then by theorem 3.4.11

$$bc = b(da) = (bd)a = 1^* a = a.$$

<u>Definition 3.4.3:</u> The cut c constructed in theorem 3.4.13 is called the quotient obtained by dividing a by b.

* Section 5. Rational Cuts and Whole Cuts

Definition 3.5.1: A cut of the form X^* is called a rational cut.

Definition 3.5.2: A cut of the form x^* is called a whole cut.

Theorem 3.5.1: If $X > Y$, or $X = Y$, or $X < Y$, then $X^* > Y^*$, or $X^* = Y^*$, or $X^* < Y^*$, and the converse is also true.

Theorem 3.5.2: (I) $(X + Y)^* = X^* + Y^*$; (II) $(X - Y)^* = X^* - Y^*$; for $X \; Y$; (III) $(XY)^* = X^*Y^*$; (IV) $\left(\frac{X}{Y}\right)^* = \frac{X^*}{Y^*}$.

Proof: (I) a) Any lower number contained in $X^* + Y^*$ is the sum of a rational number less than X and a rational number less than Y. Therefore it is also less than $X + Y$ and is also a lower number of $(X + Y)^*$.

b) Any rational number U contained in $(X + Y)^*$ is less than $X + Y$. Then since $\frac{U}{X + Y} < 1$, $U = X\frac{U}{X + Y} + Y\frac{U}{X + Y}$, U is the sum of a rational number less than X and a rational number less than Y. Therefore U is a lower number of $X^* + Y^*$. Hence,

$$(X + Y)^* = X^* + Y^*.$$

(II) If $X > Y$, then $X = (X - Y) + Y$. By part I $X^* = (X - Y)^* + Y^*$, then $(X - Y)^* = X^* - Y^*$.

(III) a) Any rational number X^*Y^* is the product of a rational number less than X and a rational number less than Y. Therefore any number less than XY is a lower number of $(XY)^*$.

b) Any lower number U contained in $(XY)^*$ is less than XY. By theorem 2.5.14, there is a rational number U_1 such that $U < U_1 < XY$. Then $\frac{U}{U_1} < 1$, and $\frac{U_1}{Y} < X$. Since $U = \frac{U_1}{Y}\left(Y\frac{U}{U_1}\right)$, U is a product of a lower number of X^* and a lower number of Y^*. Hence U is a lower number of X^*Y^*, and we have proved that $(XY)^* = X^*Y^*$.

(IV) Since $X = \frac{X}{Y}Y$, it follows by part III that

$$X^* = \left(\frac{X}{Y}\right)^*Y^*, \text{ then } \left(\frac{X}{Y}\right)^* = \frac{X^*}{Y^*}.$$

<u>Theorem 3.5.3:</u> A whole cut satisfies the five postulates of natural numbers, when 1^* is taken in the place of 1, and

$$(x^*)' = (x')^*.$$

Proof: Let W^* be the set of all whole cuts.

1) 1^* is in the set W^*

2) For x^* contained in W^*, there is an $(x^*)'$ in W^*.

3) Since $x' \neq 1$, then $(x')^* \neq 1^*$, and $(x^*)' \neq 1^*$.

4) If $(x^*)' = (y^*)'$, then $(x')^* = (y')^*$, $x' = y'$, $x = y$, and $x^* = y^*$.

5) Let N^* be a class of whole cuts such that

a) 1^* is contained in N^*.

b) If x^* is contained in N^*, then $(x^*)'$ is contained in N^*.

Let N be the class of whole numbers x for which x^* is contained in N^*. Then 1 is contained in N, and for every x contained in N, x' also is contained in N. Hence N is the set of all whole numbers, and N^* is the set of all whole cuts.

Thus, since in terms of $=, >, <$, sum, difference, product and quotient as previously defined, the rational cuts have all of the properties of rational numbers established in chapter II; in particular, whole cuts have all the properties of whole numbers.

Theorem 3.5.4: The rational numbers are those cuts for which there is a smallest upper number X. Indeed, X is then the cut.

Proof: 1) The cut X (previously X^*) has X (a rational number in the original sense) for the least upper number.

2) Given X as the least upper number in the cut a, then since every lower number is less than X, and every upper number is greater than or equal to X, the cut is X (or the old X^*).

Theorem 3.5.5: If a is a cut, then X is a lower number when X is less than a, or an upper number when X is greater than or equal to a.

<u>Theorem 3.5.6:</u> If $a < b$, there is a Z such that

$$a < Z < b.$$

<u>Theorem 3.5.7:</u> Every $Z > ab$ has the form $Z = XY$, where $X \geq a$, $Y \geq b$.

Proof: Let d be the smaller of the cuts 1 and $\frac{Z - ab}{(a + b) + 1}$, then $d \leq 1$, $d \leq \frac{Z - ab}{(a + b) + 1}$.

By theorem 3.5.6 Z_1 and Z_2 exist such that

$a < Z_1 < a + d$, $\quad b < Z_2 < b + d$. Then

$$Z_1 Z_2 < (a + d)(b + d) = (a + d)b + (a + d)d$$
$$\leq (a + d)b + (a + 1)d = ab + ((a + b) + 1)d$$
$$\leq ab + (Z - ab) = Z.$$

Now $Z = \frac{Z}{Z_2} Z_2$, and $\frac{Z}{Z_2} = Z \frac{1}{Z_2} > (Z_1 Z_2)\frac{1}{Z_2} = Z_1 > a$.

Then if we take $X = \frac{Z}{Z_2}$, and $Y = Z_2$, we have

$$Z = \left(\frac{Z}{Z_2}\right)(Z_2) = XY,$$

as was to be shown.

<u>Theorem 3.5.8:</u> For every a there is a $bb = a$.

Proof: I) There is only one solution, because if $b_1 > b_2$, then $b_1 b_1 > b_2 b_2$.

II) Consider the set of rational numbers X such that XX a. This set is a cut. For

1) If $X < 1$, and $X < a$, then $XX < X \cdot 1 = X < a$. If $X \geq 1$ and $X \geq a$, then $XX \geq X \cdot 1 = X \geq a$. Hence the set contains some numbers, but not all numbers.

2) If $XX < a$, $Y < X$, then $YY < XX < a$.

3) If $XX < a$, one can choose a Z smaller than the smaller of the cuts 1 and $\frac{a - XX}{X + (X + 1)}$. Thus $Z < 1$, $Z < \frac{a - XX}{X + (X + 1)}$; then since $X + Z > X$, and

$$(X + Z)(X + Z) = (X + Z)X + (X + Z)Z < (XX + ZX) + (X + 1)Z$$

$$= XX + (X + X + 1)Z < XX + (a - XX) = a,$$

and XX is a cut. We will call the constructed cut b.

We now prove that $bb = a$.

If $bb > a$, then by theorem 3.5.6 there is a Z such that $bb > Z > a$.

Now if Z is a lower number of a, we can write $Z = X_1X_2$, $X_1 < b$, $X_2 < b$; or if X is the larger of X_1 and X_2, $X < b$, we have $Z \leq XX < a$. But this means Z is a lower of a, which contradicts the assumption

$$bb > Z > a.$$

When $bb < a$, then there is a Z by theorem 3.5.6 such that $bb < Z < a$. Z has by theorem 3.5.7 the form $Z = X_1X_2$, $X_1 \geq b$, $X_2 \geq b$, when X is the smaller of X_1 and X_2, then $X \geq b$,

$$Z \geq XX \geq a,$$

as was given above.

<u>Definition 3.5.3</u>: Every cut which is not a rational number is called an irrational number.

<u>Theorem 3.5.9:</u> There is an irrational number.

Proof: It is sufficient to show that the b which satisfies $bb = 1'$ is irrational. Suppose that b is rational, that is that $b = \frac{x}{y}$, where x and y are chosen so that y has the smallest possible value. Then

$1' = bb = \frac{x}{y}\cdot\frac{x}{y} = \frac{xx}{yy}$. Hence $yy < 1'(yy) = xx = (1'y)y$

$< (1'y)(1'y)$, and $y < x < 1'y$.

Put $x - y = u$, then $y + u = x < 1'y = y + y$, $u < y$.

Now by direct calculation,

$$(v + w)(v + w) = (v + w)v + (v + w)w = (vv + wv) + (vw + ww)$$
$$= (vv + 1'(vw)) + ww.$$

Now if we let $y - u = t$, then

$$xx + tt = (y + u)(y + u) + tt = (yy + 1'(yu)) + (uu + tt)$$
$$= (yy + (1'u)(u + t)) + (uu + tt)$$
$$= (yy + 1'(uu)) + ((1'(ut) + uu) + tt)$$
$$= (yy + 1'(uu)) + (u + t)(u + t) = (yy + 1'(uu)) + yy$$
$$= 1'(yy) + 1'(uu) = xx + 1'(uu), \text{ then } tt = 1'(uu),$$

Hence $$\frac{t}{u}\cdot\frac{t}{u} = 1'.$$

But $u < y$, and since y was taken to be the lowest common denominator, this is a contradiction. Therefore, the theorem as stated holds.

CHAPTER IV

REAL NUMBERS

Section 1. Definition

Definition 4.1.1: Henceforth, we will call cuts positive numbers. The rational numbers and whole numbers previously considered will now be called positive rational numbers and positive whole numbers respectively.

Definition 4.1.2: There exists a number zero, written 0, different from any positive number.

Definition 4.1.3: There exist numbers, different from the positive numbers and 0, called negative numbers such that 1) for any a (that is any positive number) there is a corresponding negative number -a (read minus a),

2) if $-a = -b$, then $a = b$.

Definition 4.1.4: The totality of all positive numbers, zero, and all negative numbers will be called the set of all real numbers.

Capital letters, A, B, C, D, E, F, and G, will be used to represent real numbers.

From the definition, the usual theorems of equality are readily seen to hold.

<u>Theorem 4.1.1:</u> $A = A$.

<u>Theorem 4.1.2:</u> If $A = B$, then $B = A$.

<u>Theorem 4.1.3:</u> If $A = B$, $B = C$, then $A = C$.

Section 2. Order

Definition 4.2.1: $|A| = \begin{cases} a, & \text{when } A = a, \\ 0, & \text{when } A = 0, \\ a, & \text{when } A = -a. \end{cases}$

The number $|A|$ is called the absolute value of A.

Theorem 4.2.1: $|A|$ is positive for both positive and negative A's.

Proof by definition 4.2.1.

Definition 4.2.2: If A and B are not both positive, then $A > B$ means that one of the following cases holds:

1) A is negative, B is negative and $|A| < |B|$,

2) $A = 0$, B is negative,

3) A is positive, B is negative,

4) A is positive, B is 0.

Definition 4.2.3: If $A < B$, then $B > A$.

Theorem 4.2.2: When A and B are given exactly one of the following cases holds: $A = B$, or $A > B$, or $A < B$.

Definition 4.2.4: $A \geq B$ means that either $A > B$, or $A = B$.

Definition 4.2.5: $A \leq B$ means that either $A < B$, or $A = B$.

Theorem 4.2.3: If $A \geq B$, then $B \leq A$.

Theorem 4.2.4: The positive numbers are greater than 0; the negative numbers are less than 0.

Proof:1. By definition 4.2.2 $a > 0$.

2. Then from $A > 0$ by definition 4.2.2, it follows that A is positive, i.e. that $A = a$.

3. By theorem 4.2.2, since $-a \neq 0$, and $-a \neq b$ (any positive number) $-a < 0$.

4. Hence for any $A < 0$, by definition 4.2.3, $0 > A$, and by definition 4.2.2, A is a negative number, $A = -a$.

<u>Theorem 4.2.5:</u> $|A| \geq 0$.

<u>Theorem 4.2.6:</u> If $A < B$, $B < C$, then $A < C$.

Proof: Three cases must be considered.

1. Let $C > 0$. Then if $A > 0$, then also $B > 0$, by theorem 3.2.7. For the case $A \leq 0$, then $A < C$ by definition.

2. Let $C = 0$. Then $B < 0$, and $A < B$ implies that A is negative, $A < 0 = C$.

3. Let $C < 0$. Then $B < 0$, and $A < 0$. By definition 4.2.2

$A < B$ implies $|A| > |B|$, and $B < C$ implies $|B| > |C|$, then $|A| > |C|$, hence $A < C$.

<u>Theorem 4.2.7:</u> If $A \leq B$, $B < C$, or $A < B$, $B \leq C$, then $A < C$.

<u>Theorem 4.2.8:</u> If $A \leq B$, $B \leq C$, then $A \leq C$.

<u>Definition 4.2.6:</u> If $A \leq 0$, A is rational if $A = 0$, or for $A < 0$, if $|A|$ is rational.

<u>Definition 4.2.7:</u> When $A < 0$, A is called irrational if it is not rational.

Note that if a is irrational, then $a + X$ is irrational; if $a + X = Y$, $a = Y - X$, whence a would be rational. Thus $(a + X)$ is a positive irrational number, and $-(a + X)$ is a negative irrational number.

<u>Definition 4.2.8:</u> If $A \leq 0$, then if $A = 0$, A is called a whole number, or for $A < 0$, if $|A|$ is a whole number, then A is called a whole number.

<u>Theorem 4.2.9:</u> Every whole number is rational.

Section 3. Addition

Definition 4.3.1:

$$A + B = \begin{cases} -(|A| + |B|), & \text{when } A < 0,\ B < 0; \\ \left.\begin{matrix} |A| - |B| \\ 0 \\ -(|B| - |A|) \end{matrix}\right\}, & \text{when } A > 0,\ B < 0, \ \begin{cases} |A| > |B|; \\ |A| = |B|; \\ |A| < |B|; \end{cases} \\ B + A, & \text{when } A < 0,\ B > 0; \\ B, & \text{when } A = 0; \\ A, & \text{when } B = 0. \end{cases}$$

Theorem 4.3.1: $A + B = B + A.$

Definition 4.3.2:

$$-A = \begin{cases} 0 \text{ for } A = 0, \\ |A| \text{ for } A < 0. \end{cases}$$

Theorem 4.3.2: If $A > 0$, $A = 0$, or $A < 0$, then it follows respectively that $-A < 0$, $-A = 0$, or $-A > 0$.

Proof by definitions 4.1.3, and 4.3.2.

Theorem 4.3.3: $-(-A) = A.$

Proof by definitions 4.1.3, 4.2.1, and 4.3.2.

Theorem 4.3.4: $|-A| = |A|.$

Proof by definitions 4.1.3, 4.2.1, and 4.3.2.

Theorem 4.3.5: $A + (-A) = 0.$

Proof by definitions 4.3.1 and 4.3.2, and theorem 4.3.4.

Theorem 4.3.6: $-(A + B) = -A + (-B).$

Proof: By theorem 4.3.1 $-(A + B) = -(B + A)$ and $-A + (-B) = -B + (-A)$. We may assume without loss of generality that $A \geq B$.

1. When $A > 0$, $B > 0$, then $-A + (-B) = -(A + B)$,

2. When $A > 0$, $B = 0$, then $-A + (-B) = -A + 0 = -A$

$$= -(A + 0) = -(A + B),$$

3. When $A > 0$, $B < 0$, if

   i. $A > |B|$, then $A + B = A - |B|$, $-A + (-B) = -A - |B|$

   $$= -(A - |B|) = -(A + B),$$

   ii. $A = |B|$, then $A + B = 0$, $-A + (-B) = -A + |B|$

   $$= 0 = -(A + B),$$

   iii. $A < |B|$, then $A + B = -(|B| - A)$,

   $$-A + (-B) = -A + |B| = |B| - A = -(A + B).$$

4. When $A = 0$ then $-A + (-B) = 0 + (-B) = -B = -(0 + B)$

$$= -(A + B).$$

5. When $A < 0$, $B < 0$ then $A + B = -(|A| - |B|)$,

$$-A + (-B) = |A| - |B| = -(A + B).$$

<u>Definition 4.3.3</u>: $A - B = A + (-B)$.

<u>Theorem 4.3.7</u>: $-(A - B) = B - A$.

Proof: By theorems 4.3.6 and 4.3.3

$$-(A - B) = -(A + (-B)) = -A + (-(-B)) = -A + B = B + (-A)$$
$$= B - A.$$

<u>Theorem 4.3.8</u>: If $A - B > 0$, $A - B = 0$, or $A - B < 0$, then $A > B$, $A = B$, or $A < B$, respectively.

Proof: Because $-B$ is a real number, one can write $-B$ in the place of B for the corresponding case of

$A - B > 0$, $A - B = 0$, or $A - B < 0$, and

$A > -B$, $A = -B$, $A < -B$.

In the case of $A = 0$, or $B = 0$ the assertion is evident.

For $A > 0$ and $B > 0$ we have

$$A - B = A - (-B) = \begin{cases} 0 \text{ if } |A| > |-B| \\ 0 \text{ if } |A| = |-B| \\ 0 \text{ if } |A| < |-B| \end{cases}$$

By definition 4.3.1 the other cases can be handled in a similar manner.

Theorem 4.3.9: If $A > B$, $A = B$, or $A < B$, then it follows respectively that $-A < -B$, $-A = -B$, or $-A > -B$.

Theorem 4.3.10: Every real number may be represented as the difference of two positive numbers.

Proof: 1) If $A > 0$, then $A = (A + 1) - 1$.

2) If $A = 0$, then $A = 1 - 1$.

3) If $A < 0$, then $-A = |A| = (|A| + 1) - 1$,

$$A = -((|A| + 1) - 1) = 1 - (|A| + 1).$$

Theorem 4.3.11: If $A = a_1 - a_2$, $B = b_1 - b_2$, then $A + B = (a_1 + b_1) - (a_2 + b_2)$

Proof: 1) Let $A > 0$, $B > 0$, then for any positive numbers a, b, c, d, we have

$$(a + b) + (c + d) = (a + b) + (d + c) = ((a + b) + d) + c$$
$$= c + (a + (b + d)) = (c + a) + (b + d),$$

then $(A + B) + (a_2 + b_2) = a_1 + b_1$.

2) Let $A < 0$, $B < 0$, then by theorem 4.3.7

$a_2 - a_1 = - A > 0$, $b_2 - b_1 = -B > 0$.

Then by part 1,

$$-A + (-B) = (a_2 + b_2) - (a_1 + b_1)$$

$$A + B = -(-A + (-B)) = (a_1 + b_1) - (a_2 + b_2).$$

3) Let $A > 0$, $B < 0$,

then $a_1 - a_2 > 0$, $b_2 - b_1 > 0$.

A) Let $A > |B|$, then $a_1 - a_2 > b_2 - b_1$.

Since $a_1 - b_1 = (a_1 - a_2) + a_2) + b_1$

$= (a_1 - a_2) + (a_2 - b_1) = (a_2 + b_1) + (a_1 - a_2)$

$= (a_2 + b_1) + ((b_2 - b_1) + ((a_2 - a_1) - (b_2 - b_1)))$

$= ((a_2 + b_1) + (b_2 - b_1)) + ((a_1 - a_2) - (b_2 - b_1))$

$= (a_2 + (b_1 + (b_2 - b_1))) + ((a_1 - a_2) - (b_2 - b_1))$

$= (a_2 + b_2) + ((a_1 - a_2) - (b_2 - b_1)),$

then $(a_1 + b_1) - (a_2 + b_2)$

$= (a_2 + b_2) + (a_1 - a_2) - (b_2 - b_1) + (a_2 - b_2)$

$= (a_1 - a_2) - (b_2 - b_1) = A - B = A + B.$

B) Let $A < |B|$.

Then from part A) $A + B = -(-B + (-A))$

$= -((b_2 - b_1) + (a_2 - a_1)) = -((b_2 + a_2) - (b_1 + a_1))$

$= (b_1 + a_1) - (b_2 + a_2) = (a_1 + b_1) - (a_2 + b_2)$

C) Let $A = |B|$.

Then $a_1 - a_2 = b_2 - b_1$,

$$a_1 = a_2 + (b_2 - b_1),$$

$$a_1 + b_1 = a_2 + b_2,$$

$$A + B = 0 = (a_1 + b_1) - (a_2 + b_2).$$

4) Let $A < 0$, $B > 0$.

Then by part 3)

$B + A = (b_1 + a_1) - (b_2 + a_2)$

$A + B = (a_1 + b_1) - (a_2 + b_2)$.

5) Let $A = 0$.

Then $a_1 = a_2$, and $A - B = B$.

A) Let $b_1 > b_2$, then

$(b_1 - b_2) + (a_1 - b_2) = ((b_1 - b_2) + b_2) + a_1$

$= b_1 + a_1 = a_1 + b_1$

$B = (b_1 - b_2) = (a_1 + b_1) - (a_1 + b_2)$

$= (a_1 + b_1) - (a_2 + b_2)$.

B) Let $b_1 = b_2$.

Then $B = 0 = (a_1 + b_1) - (a_2 + b_2)$.

C) Let $b_1 < b_2$.

Then by part A)

$-B = (b_2 - b_1) = (a_2 + b_2) - (a_1 + b_1)$,

$B = -(-B) = (a_1 + b_1) - (a_2 + b_2)$.

6) Let $B = 0$. Then $A + B = (a_1 + b_1) - (a_2 + b_2)$.

<u>Theorem 4.3.12</u>: $(A + B) + C = A + (B + C)$.

Proof: By theorem 4.3.10; $A = a_1 - a_2$, $B = b_1 - b_2$, $C = c_1 - c_2$. By theorem 4.3.11

$(A + B) + C = ((a_1 + b_1) - (a_2 + b_2) + (c_1 - c_2)$

$= ((a_1 + b_1) + c_1) - ((a_2 + b_2) + c_2)$

$= (a_1 + (b_1 + c_1)) - (a_2 + (b_2 + c_2))$

$= (a_1 - a_2) + ((b_1 + c_1) - (b_2 + c_2)) = A + (B + C).$

<u>Theorem 4.3.13:</u> For any given A and B, there is a unique C such that $B + C = A$, or $C = A - B$.

Proof: That $C = A - B$ is a solution of $B + C = A$ follows from $B + (A - B) = (A - B) + B = (A + (-B)) - B$

$= A + (-B + B) = A + 0 = A.$

From $B + C = A$ it follows that $A - B = A + (-B)$

$= -B + A = -B + (B + C) = (-B + B) + C = 0 + C = C,$

hence C is unique.

<u>Theorem 4.3.14:</u> If $A + C > B + C$, or $A + C = B + C$, or $A + C < B + C$, then it follows respectively that

$A > B$, or $A = B$, or $A < B$.

Proof: The given relations are equivalent respectively to $(A + C) - (B + C) > 0$, or $(A + C) - (B + C) = 0$, or $(A + C) - (B + C) \; 0$, from which follows respectively $(A - B) > 0$, or $(A - B) = 0$, or $(A - B) < 0$, or

$A > B$, or $A = B$, or $A < B$.

<u>Theorem 4.3.15:</u> If $A > B$, $C > D$, then $A + C > B + D$.

<u>Theorem 4.3.16:</u> If $A \geq B$, $C > D$, or $A > B$, $C \geq D$, then

$A + C > B + D.$

<u>Theorem 4.3.17:</u> If $A \geq B$, $C \geq D$, then $A + C \geq B + D$.

Section 4. Multiplication

Definition 4.4.1: $A \cdot B$ $\begin{cases} -(|A| \cdot |B|), & \text{when } A > 0,\ B < 0, \text{or } A<0, B>0; \\ |A| \cdot |B|, & \text{when } A < 0,\ B < 0, \text{ or } A > 0,\ B > 0; \\ 0, & \text{when either } A = 0, \text{ or } B = 0. \end{cases}$

Theorem 4.4.1: If $AB = 0$, then either A or B is zero.

Theorem 4.4.2: $|AB| = |A| \cdot |B|$.

Theorem 4.4.3: $AB = BA$.

Theorem 4.4.4: $A \cdot 1 = A$.

Theorem 4.4.5: If $A \neq 0$, $B \neq 0$, then $AB = |A| \cdot |B|$, if neither or both of A and B are negative, and $AB = -|A| \cdot |B|$ if either A or B (but not both) is negative.

Theorem 4.4.6: $(-A)B = A(-B) = -(AB)$.

Proof: 1) If either $A = 0$, or $B = 0$, then all three expressions are zero.

2) If $A \neq 0$, $B \neq 0$, then by theorem 4.4.2, all three expressions have the same absolute value, A B , and by theorem 4.4.5, all three are positive, or negative depending upon whether exactly one, or none or two of the numbers A, B are negative.

Theorem 4.4.7: $(-A)(-B) = AB$.

Proof: By theorem 4.4.6, we have $(-A)(-B) = A(-(-B)$

$$= AB.$$

Theorem 4.4.8: $(AB)C = A(BC)$.

Proof: 1) If one of A, B, C is zero, then each side of the expression is zero.

2) If neither A, B, nor C is zero, then by theorem 4.4.2 each side has the same absolute value, and by theorem 4.4.5 both sides are positive are negative together.

<u>Theorem 4.4.9:</u> $a(b - c) = ab - ac$.

<u>Theorem 4.4.10:</u> $A(B + C) = AB + AC$.

Proof: 1) Let $A > 0$. By theorem 4.3.10, $B = b_1 - b_2$, $C = c_1 - c_2$. By theorem 4.3.11 $B - C = (b_1 - c_1) - (b_2 - c_2)$ and by theorems 4.4.9 and 3.4.4

$$A(B + C) = (Ab_1 - Ab_2) + (Ac_1 - Ac_2) = A(b_1 - b_2) + A(c_1 - c_2)$$
$$= AB + AC.$$

2) If $A = 0$, then $A(B + C) = 0 = AB + AC$.

3) If $A < 0$, then by part 1, $(-A)(B + C) = (-A)B + (-A)C$, then $-(A(B + C)) = -((-A)B + (-A)C)$, and

$$A(B + C) = -((-A)B + (-A)C = -((-A)B + (-((-A)C)) = AB + AC.$$

<u>Theorem 4.4.11:</u> $A(B - C) = AB - AC$.

<u>Theorem 4.4.12:</u> If $A > B$, then for $C > 0$, $C = 0$, or $C < 0$, it follows respectively that $AC > BC$, or $AC = BC = 0$, or

$$AC < BC.$$

Proof: Since $A - B > 0$, then $(A - B)C > 0$, or $(A - B)C = 0$, or $(A - B)C < 0$, as $C > 0$, or $C = 0$, or $C < 0$. Then by theorem 4.4.11 $(A - B)C = C(A - B) = CA - BC = AC - BC$, it follows by theorem 4.3.8 that

$AC > BC$, or $\quad AC = BC$, or $\quad AC < BC$.

<u>Theorem 4.4.13:</u> For the equality $BC = A$, where A and B are given and B does not equal to zero, there is one and only one solution C.

Proof: 1) There is only one ~~solution~~, because if

$BC_1 = A = BC_2$, then $0 = BC_1 - BC_2 = B(C_1 - C_2)$

then by theorem 4.4.1 $0 = (C_1 - C_2)$, $C_1 = C_2$.

2) a) Let $B > 0$, then $C = \frac{1}{B}A$ is a solution, because

$BC = B(\frac{1}{B} A) = (B\frac{1}{B})A = 1 A = A$.

b) Let $B < 0$. Then $C = -(\frac{1}{|B|}A)$ is a solution, because

$A = |B| (\frac{1}{|B|} A) = |B| (-C) = (-|B|)C = BC$.

<u>Definition 4.4.2:</u> The C constructed in theorem 4.4.13 is call the quotient obtained by dividing A by B.

Section 5. Dedekind's Principal Theorem

Theorem 4.5.1: Let therebe given any separation of all real numbers into two classes with the following properties:

1) There is a number in the first class and a number in the second class.

2) Every number in the first class is less than every number in the second class.

Then ther exists exactly one real number A such that every B A belongs to the first class, and every B A belongs to the second class.

Proof: A) If such a number A exists, it is unique; for suppose A_1 and A_2 satisfied the theorem, with $A_1 < A_2$. Then from

$$(1 + 1)A_1 = A_1 + A_1 < A_1 + A_2 < A_2 + A_2 = (1 + 1)A_2,$$

we have

$$A_1 < \frac{A_1 + A_2}{1 + 1} < A_2,$$

whence the number $\frac{A_1 + A_2}{1 + 1}$ would be in both classes, contrary to condition 2).

B) To prove the existence of A, four cases must be considered:

1. Suppose there is a positive number in the first class. Consider the out constructed as follows.

Every positive rational number in the first class, except the possibly greatest rational number in the first class, belongs to the lower class; all other positive rational numbers (that is the possibly greatest rational number of the first class, and all those in the second class) belong to the upper class. Now, this separation is a cut, because

i) Since the first class contains a positive number, it contains all smaller positive rational numbers (there are such numbers by theorem 3.5.5) and also contains a number such that there is a larger positive rational number in the first class. Thus the lower class is not void.

Since the second class contains a number, it contains all greater positive rational numbers, there are such numbers by theorem 3.5.5. Hence the upper class also is not void.

ii) Every number of the lower class is less than every number of the upper class, because every number of the first class is less than every number of the second class, and the possibly greatest positive rational number of the first class is certainly greater than every number of the lower class.

iii) The lower class contains no greatest positive rational number, because either the first class contains no greatest positive rational number, or if it does, this number by the construction was placed in the upper class, and then theorem 2.5.14 assures us that there is no greatest positive rational number among all the rational numbers less than any particular such number.

Then by i), ii), and iii) this separation is a cut. Let A be the positive number defined by this cut. Then A satisfies the statement of the theorem, because

a) Suppose there is given a B such that $B < A$. Choose according to theorem 3.5.6 (with $a = B$, $b = A$ if $B > 0$, or with $a = \frac{A}{1+1}$, $b = A$ if $B \leq 0$) a positive rational number Z such that $B < Z < A$. Then Z is a lower number for the cut A, and thus belongs to the first class. Hence B belongs to the first class

b) Suppose there is a given B such that B A. Choose by theorem 3.5.6, a Z such that $A > Z > B$. Then Z is an upper number for the cut A and by theorem 3.5.6 thus belongs to the second class. Hence B belongs to the second class.

2) Suppose that every positive number in the

second class, and zero is in the first class. Then every negative number lies in the first class, and $A = 0$ satisfies the statement of the theorem.

3) Suppose that 0 lies in the second class, and every negative number lies in the first class. Then every positive number lies in the second class, and again $A = 0$ satisfies the statement of the theorem.

4) Suppose there is a negative number in the second class. Consider the following separation: B is in the new first class if $-B$ was in the original second class; B is in the new second class if $-B$ was in the original first class. This separation satisfies the conditions of the hypothesis because

i) there is a number in each class,

ii) from $B_1 < B_2$ it follows by theorem 4.3.9 that $-B_2 < -B_1$.

Moreover, this new separation comes under case 1), since there is a positive number in the new first class. By case 1) then, there exists a number A_1 such that every $B < A_1$ lies in the new first class and every $B > A_1$ lies in the new second class. Put, then, $-A_1 = A$, and from $B < A$ or $B > A$ it follows respectively that $-B > A_1$ or $-B < A_1$. Thus if $-B$ lies in the new second class or in the new first class, then B lies respectively in the original first class or in the orginal second class.

In clossing, it is to be noted that every real number A gives rise to exactly two such separations; namely one with $B \leq A$ as the first class and $B > A$ as the second class; and the other with $B < A$ as the first class and $B \geq A$ as the second class.

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