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Detection and Estimation of a Mixture in Power Law Processes for a Repairable System

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The power law process has proved to be a useful tool in characterizing the failure process of repairable systems. This paper presents a procedure for detecting and estimating a mixture of reliable and unreliable (defective) systems. The test of a mixture, based on a simple likelihood ratio, is illustrated with truncated failure data for copy machines. Bootstrap methods are used to gauge the estimation uncertainty, and optimal decisions for system replacement are determined based on the observed likelihood.

Key Words: Bootstrap Sampling; EM Algorithm; Failure Truncation; Maximum Likelihood; Minimal Repair; Warranty.

For a repairable system, it is crucial to know not only whether it is reliable at the start of operation but also how the reliability changes over usage time. A system must be taken out of operation before repairs become too frequent and operation costs soar. While some systems will show age only after experiencing a great amount of usage time, other systems may be prone to frequent failures from the start. For industries that manufacture complex systems that are susceptible to failures from competing risks (i.e., risks of failure from different sources), it is not uncommon to find a heterogenous population with a majority of reliable systems mixed in with a small fraction of defective ones.

We refer to these good systems as "conforming" because the quality might be measured not only in terms of reliability (e.g., time on test) but in other aspects having to do with system operation. The "nonconforming" systems will exhibit shorter operation time between repairs, and unlike a repair to the nondefective conforming systems, these repairs can include different failure modes that are seemingly unrelated.

In the automotive manufacturing industry, for example, the small proportion of new cars that make repeat trips to the repair shop are called *lemons*, and several states have adopted consumer protection rights ("lemon laws") that will force the manufacturer to replace the defective product with no cost to the consumer. There is an industry of law practices just for lemon law cases, as pointed out in Lehto (2000) and Megna (2003).

By treating the defective products as a contaminated subpopulation, the time to failure of a new item can be described with a *mixture distribution*; if T is the product lifetime, then its lifetime distribution, $F(t)$, is extricated to

$$F(t) = \omega F_a(t) + (1-\omega)F_0(t), \quad (1)$$

where F_0 is the lifetime distribution of the normal (nondefective) products, ω is the proportion of defective (or nonconforming) products that have distribution F_a, where $F_a(t) > F_0(t)$. Here "lifetime" refers to time to failure *after* the most recent re-

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pair, so a system that is repaired four times can have five distinct lifetimes. We refer to the aggregate of these between-failure times as the system's "total lifetime".

Manufacturers of large, repairable systems, including the automobile industry, can benefit greatly by quickly identifying a finished product that was generated from the nonconforming population F_a and getting it out of service as soon as possible. In general, the defective items are costly to the manufacturer, thereby greatly influencing the warranty policy and limiting the protection the manufacturer will offer to the consumer. Mixtures have been helpful in modeling repair times for warranty policy, including heuristic models by Majeske and Herrin (1995). Majeske (2003) used a mixture hazard function to model the time to first warranty claim and estimated the fraction of vehicles containing a manufacturing or assembly defect when leaving the assembly plant.

In this paper, the repair process is modeled as a *minimal repair process* generated from the mixture in Equation (1). Once the system fails, it is automatically repaired to be as good as an identical system that has survived to the same age. The resulting sequence of failure times constitutes a nonhomogeneous Poisson process with mean rate function equal to the underlying cumulative hazard rate. Obviously, if the system has a greatly increasing rate of failure, the overall cost of operating the system is strongly dependent on the replacement policy. Kvam, Singh, and Whitaker (2002) considered estimating the system lifetime distribution in the case the system was known to have an increasing failure rate.

For practical consideration, we focus on the nonhomogeneous Poisson process with intensity function

$$v(t) = \frac{\beta}{\theta}(t/\theta)^{\beta-1}, \qquad (2)$$

which is commonly accepted as an effective model for many repairable systems, e.g., see Rigdon and Basu (2000). A convenient alternative parameterization for Equation (2) is

$$v(t) = \lambda\beta t^{\beta-1}. \qquad (3)$$

This model is called the *power law process* (PLP) because the intensity function is proportional to a power of t. We call λ the intensity parameter, β the shape parameter, and θ the scale parameter. The power law process is frequently used to model repairable system lifetimes, as evident in Duane (1964), Ridgon et al. (1998), and Ridgon and Basu (1989). Engelhardt and Bain (1987) used a compound power law model to characterize the heterogeneity of different systems by treating λ as a random variable from the gamma distribution. This frailty-type model accounts for general heterogeneity of the population, but is not effective in modeling nonconforming systems. In this paper, we choose to model multiple systems as mixture power law processes with two point mixture distributions. These correspond to two types of intensity functions, $v_0(t)$ and $v_a(t)$ for conforming and nonconforming systems, respectively. The higher failure rate of the nonconforming subpopulation is characterized by an inequality between their respective intensity parameters: $\lambda_a > \lambda_0$.

Consider n manufactured systems with intensity function $v_i(t) = \lambda_i\beta_i t^{\beta_i - 1}$, $i = 1, \ldots, n$. The systems are possibly time truncated or failure truncated. For time-truncated systems, we observe system i over time interval $(0, \tau_i)$; τ_i may be the current calendar time. Denote t_{ij} as the jth failure time for system i, and $j = 1, \ldots, k_i$, where k_i is the number of failures before censoring time τ_i.

For the failure-truncated case, a system is taken off test after a fixed number of failures is observed. Denote k_i as the pre-fixed number of failures, then the failure times, t_{ij}'s, are recorded for $j = 1, \ldots, k_i$. In the example that follows, the data can be time truncated or failure truncated. The detection of a PLP is shown in Section 2 by using copy-machine failure times as an example. The copy machines exhibit a PLP mixture of two intensity parameters (and a single shape parameter, β). Estimation, based on maximum likelihood, is described in Section 3. In Section 4, we use these estimates to develop an optimal strategy for warranty decision making.

Exploratory Study of Copy-Machine Failure

Figure 1 shows the failure-time data for a group of 20 copy machines (Zaino and Berke (1992)). For these machines, time is measured by the number of *actuations*, i.e., the number of copies made, and the time at installation is defined to be 0. This data set (adjusted for staggered installation times) is displayed in Table 1. Copiers removed from the test upon 8 failures were failure truncated, while other copiers are regarded as time censored at $\tau = 40{,}000$ actuations.

For failure time T_i and number of failures K_i, we

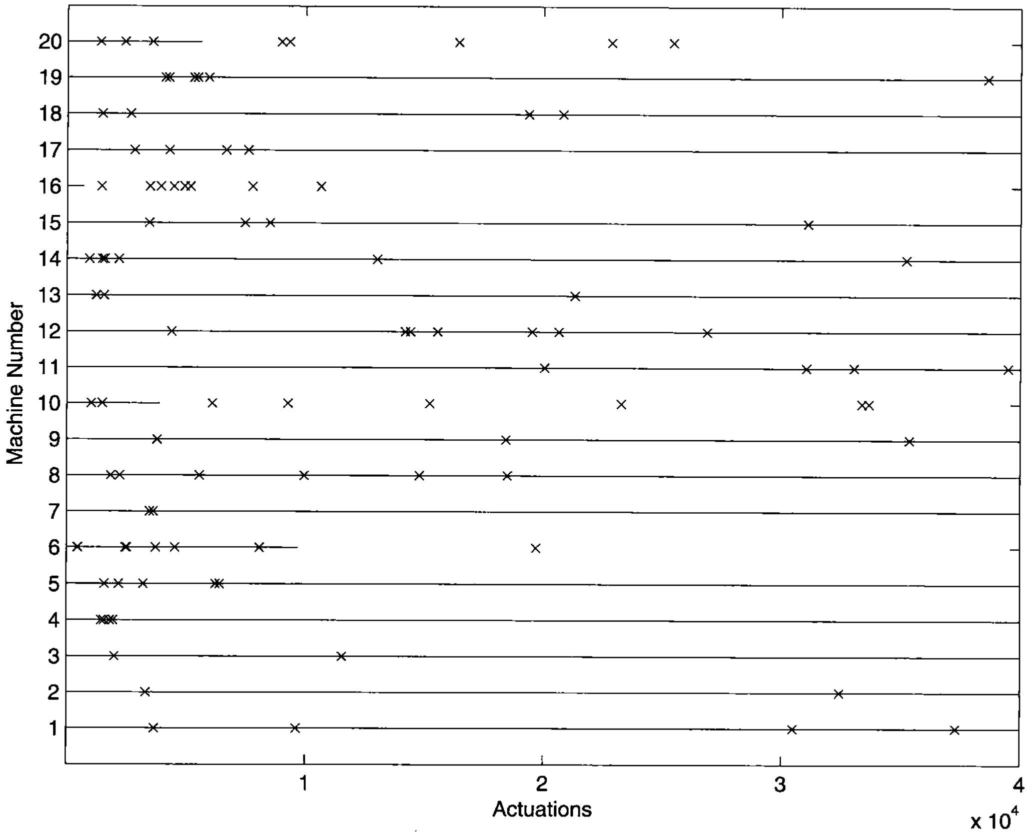

FIGURE 1. Number of Actuations Between Failures for 20 Tested Copy Machines. Data from Zaino and Berke (1992).

use the notation of Rigdon et al. (1998) for cases where some systems are failure truncated and others are time truncated:

$$T_i = \begin{cases} \tau_i & \text{if system } i \text{ is time truncated} \\ t_{i,k_i} & \text{if system } i \text{ is failure truncated} \end{cases}$$

$$K_i = \begin{cases} k_i & \text{if system } i \text{ is time truncated} \\ k_i - 1 & \text{if system } i \text{ is failure truncated.} \end{cases}$$

Based on the likelihood for an individual system,

$$L(\lambda_i, \beta_i) \propto \exp(-\lambda_i T_i^{\beta}) \prod_{j=1}^{k_i} \lambda_i \beta_i t_{ij}^{\beta_i - 1}, \quad (4)$$

the maximum likelihood estimators (MLEs) $\hat{\lambda}_i$ and $\hat{\beta}_i$ can be obtained as

$$\hat{\beta}_i = \frac{k_i}{\sum_{j=1}^{k_i} \log(T_i/t_{ij})}, \quad \hat{\lambda}_i = \frac{T_i}{k_i^{1/\hat{\beta}_i}}. \quad (5)$$

To obtain a more parsimonious model, we test equality of the intensity functions for individual systems. The shape parameter, β, demonstrates the reliability development efforts, i.e., $\beta > 1$ shows system reliability decreasing in time and $\beta < 1$ shows reliability growth. With the MLE $\hat{\beta}_i$ from Equation (5), it is well known (Chapter 4 of Rigdon and Basu (2000)) that the conditional distributions of the variables $2k_i\beta_i/\hat{\beta}_i$, $i = 1, \ldots, n$, given $k_1, \ldots, k_n$, are independent and chi squared with $2K_i$ degrees of freedom. The $100(1-\alpha)\%$ confidence intervals for β_i's are given as

$$\left(\frac{\chi^2_{\alpha/2}(2K_i)\hat{\beta}_i}{2k_i}, \frac{\chi^2_{1-\alpha/2}(2K_i)\hat{\beta}_i}{2k_i} \right),$$

where $\chi^2_{1-\alpha/2}(2K_i)$, and $\chi^2_{\alpha/2}(2K_i)$ are the $1-\alpha/2$ and $\alpha/2$ quantiles for chi-square distribution with $2K_i$ degrees of freedom.

The hypothesis $\beta_i = \beta$ implies that reliability development efforts are equally effective for systems being tested. Crow (1974) suggests a likelihood ratio test for testing the equality of β's,

$$H_0: \ \beta_1 = \beta_2 = \cdots = \beta_n,$$

against the alternative that at least two of the β's

TABLE 1. Number of Actuations until Failure for Copy-Machine Failure Data

i	t_{i1}	t_{i2}	t_{i3}	t_{i4}	t_{i5}	t_{i6}	t_{i7}	t_{i8}
1	3678	9619	30497	37308	—	—	—	—
2	3328	32456	—	—	—	—	—	—
3	2016	11551	—	—	—	—	—	—
4	1463	1570	1820	1956	—	—	—	—
5	1596	2189	3219	6233	6409	—	—	—
6	452	472	2467	2517	3727	4537	8079	19694
7	3487	3635	—	—	—	—	—	—
8	1847	2230	5557	9958	14795	18494	—	—
9	3783	3787	18436	35375	—	—	—	—
10	1027	1483	6101	9269	15225	23273	33389	33675
11	20057	31058	33061	39497	—	—	—	—
12	4390	14190	14420	15550	19535	20650	26890	—
13	1233	1555	21318	—	—	—	—	—
14	940	1479	1583	2177	13004	35241	—	—
15	3439	7451	8503	31126	—	—	—	—
16	1443	3464	3926	4473	4918	5161	7768	10649
17	2818	4276	6656	7581	—	—	—	—
18	1474	2653	19378	20816	—	—	—	—
19	4105	4247	5305	5466	5924	38635	—	—
20	1382	2409	3557	8974	9312	16429	22850	25455

are different based on

$$\mathbf{LR} = \sum_i \beta^* - \sum_i k_i \log \hat{\beta}_i,$$

where $\hat{\beta}_i$ is the MLE for β_i and β^* is the weighted mean of the $\hat{\beta}_1, \ldots, \hat{\beta}_n$:

$$\beta^* = \frac{\sum_{i=1}^n k_i}{\sum_{i=1}^n k_i/\hat{\beta}_i}.$$

Using an approximation similar to the Bartlett's statistical testing for equal variances in independent normal distributions, the null distribution for the test statistic $-2 \times \mathbf{LR}/a$ is $\chi^2(n-1)$, where $a = 1 + (\sum_{i=1}^n 1/k_i - 1/(\sum_{i=1}^n k_i))/(6(k-1))$. This test is applied for the copy-machine failure data in Table 1, with p-value $= 0.59$; there is no strong evidence for modeling the shape parameters differently.

Given the shape parameter β is identical for all systems, we can proceed to test the equality of λ_i's. Under $H_0 : \lambda_i = \lambda$, the likelihood function is

$$L(\lambda, \beta) \propto \prod_{i=1}^n \left\{ \exp(-\lambda_i T_i^\beta) \prod_{j=1}^{k_i} \lambda_i \beta t_{ij}^{\beta-1} \right\}.$$

The MLEs for λ_i and β satisfy the following estimating equations:

$$\lambda_i = \frac{\sum_{i=1}^n k_i}{\sum_{i=1}^n T_i^\beta}$$

$$\frac{\sum_{i=1}^n k_i}{\beta} = \left\{ \sum_{i=1}^n \lambda_i T_i^\beta \log(T_i) - \sum_{i=1}^n \sum_{j=1}^{k_i} \log(t_{ij}) \right\}.$$

If all the n systems are time truncated at τ, then β is solved explicitly as

$$\hat{\beta} = \frac{\sum_{i=1}^n k_i}{\sum_{i=1}^n \sum_{j=1}^{k_i} \log(\frac{\tau}{t_{ij}})}.$$

In other cases, explicit solutions for β and λ_i are not guaranteed.

Lee (1980) proposed a test for comparing rates of several independent PLP processes. A test can be constructed based on the count data k_i when β_i are assumed to be the same. Conditional on the total number of failure times $k = \sum_{i=1}^n k_i$, the distribution of the failures counts K_i is multinomial, with cell probabilities

$$\pi_i = \frac{\lambda_i \tau_i^\beta}{\sum_{i=1}^n \lambda_i \tau_i^\beta}, \tag{6}$$

and the problem is reduced to testing multinomial parameters with $H_0 : \pi_1 = \pi_2 = ... = \pi_n$ (versus H_a : some π_i are not equal) on the simplex $\sum \pi_i = 1$. Let β_n be a consistent estimator of β (this is explained in the next section). A test for homogeneity based on Equation (6) can be constructed from

$$\hat{\pi}_i = \frac{\tau_i^{\beta_n}}{\sum_{i=1}^n \tau_i^{\beta_n}},$$

with corresponding dispersion statistic

$$q_n = \sum_{i=1}^n \frac{(k_i - k\hat{\pi}_i)^2}{k\hat{\pi}_i}.$$

Under H_0, q_n has a limiting χ^2 distribution with $n-1$ degrees of freedom. For the copy-machine data, the test statistic q_n is calculated to be 33.84, and the hypothesis of homogeneity for λ is rejected with a p-value 0.019.

A graphical plot can be applied to detect the heterogeneity in the intensity parameters if the number of failures is large enough. Conditional on β and λ_i, the total number of failures for the ith system, K_i, is a Poisson random variable with mean $\lambda_i T_i^\beta$. Then, if K_i is sufficiently large, the transformed count data

$$Z_i = \frac{K_i - \lambda T_i^\beta}{\sqrt{\lambda T_i^\beta}}, \tag{7}$$

is approximately distributed as a standard normal distribution under H_0, where $\lambda_i = \lambda$. Hence, after replacing λ and β by their consistent estimators, a normal plot for Z_i can be used to examine the homogeneity for λ_i. To show how this procedure works, we use a simple PLP simulation below.

Simulation Example 1

We simulate a mixture power law process with $n = 200$ systems, using the simulation procedure given in Meeker and Escobar (1998, p. 418). The proportion for nonconforming systems ω is set to be 0.05; the intensity parameters for conforming and nonconforming systems are $\lambda_0 = 1$, $\lambda_a = 5$, respectively. The common shape parameter β is chosen to be 1.5, indicating reliability deterioration and the censoring time, $\tau_i = \tau = 4$, is the same for all the systems.

Figure 2(a) shows the normal plots of the transformed Z_i's for the mixture population in the simulation, where a lack of fit can be detected visually. The normal plot for the conforming systems is shown in Figure 2(b), which has no strong visual evidence for lack of fit.

Mixture Model

After testing the copy-machine failure data for goodness of fit, we assume the regular population is related to the nonconforming population via a common shape parameter for the joint processes modeled with intensity functions $v_0(t) = \lambda_0 \beta t^{\beta-1}$ and $v_a(t) = \lambda_a \beta t^{\beta-1}$. We next consider the mixture model to describe these two subpopulations.

PLP Likelihood for Mixture

The likelihood based on the failure data $\{t_{ij}, 1 \le i \le m$ and $1 \le j \le k_i\}$ is a function of the parameters of the PLP intensity function, $\{\lambda_0, \lambda_a, \beta\}$. That is, the shape parameter β is the same for both $v_0(t)$ and $v_a(t)$. The mixing parameter ω is the proportion of nonconforming items in the population and is assumed to be small ($\omega \ll 0.5$). The intensity functions for conforming and nonconforming systems are $v_0(t) = \lambda_0 \beta t^{\beta-1}$ and $v_a(t) = \lambda_a \beta t^{\beta-1}$, respectively. Then the likelihood function is

$$L(\boldsymbol{\theta}; t) \propto \prod_{i=1}^n \left\{ (1-\omega)\lambda_0^{k_i}\beta^{k_i} \exp(-\lambda_0 \tau_i^\beta) \prod_{j=1}^{k_i} t_{ij}^{\beta-1} + \omega \lambda_a^{k_i} \beta^{k_i} \exp(-\lambda_a \tau_i^\beta) \prod_{j=1}^{k_i} t_{ij}^{\beta-1} \right\}, \tag{8}$$

where $t = \{t_{ij}\}$, $i = 1, \ldots, n$, $j = 1, \ldots, k_i$, and $\boldsymbol{\theta} = \{\lambda_0, \lambda_a, \omega, \beta\}$.

Obviously, there is no general closed-form solution in Equation (8) for the MLE of $\boldsymbol{\theta}$. To set up a simple iterative method for solving the MLE, the EM algorithm (see McLachlan and Krishnan (1996), for example) can be applied by defining the unobserved quantity z_i, where $z_i = 0$ if the ith system is from the conforming population ($z_i = 1$ otherwise), so that $P(Z_i = 1) = \omega$.

With $z = \{z_i, i = 1, \ldots, m\}$, the "full data" likelihood (including z) is relatively simple and well behaved:

$$L(\boldsymbol{\theta}; t, z) \propto \prod_{i=1}^n \left\{ \lambda_0^{k_i}\beta^{k_i} \exp(-\lambda_0 \tau_i^\beta) \prod_{j=1}^{k_i} t_{ij}^{\beta-1} \right\}^{1-z_i} \times \left\{ \lambda_a^{k_i}\beta^{k_i} \exp(-\lambda_a \tau_i^\beta) \prod_{j=1}^{k_i} t_{ij}^{\beta-1} \right\}^{z_i}. \tag{9}$$

The EM algorithm solves for the MLE by estimating z (or a function of z determined through the log likelihood) and maximizing over the simpler likelihood in

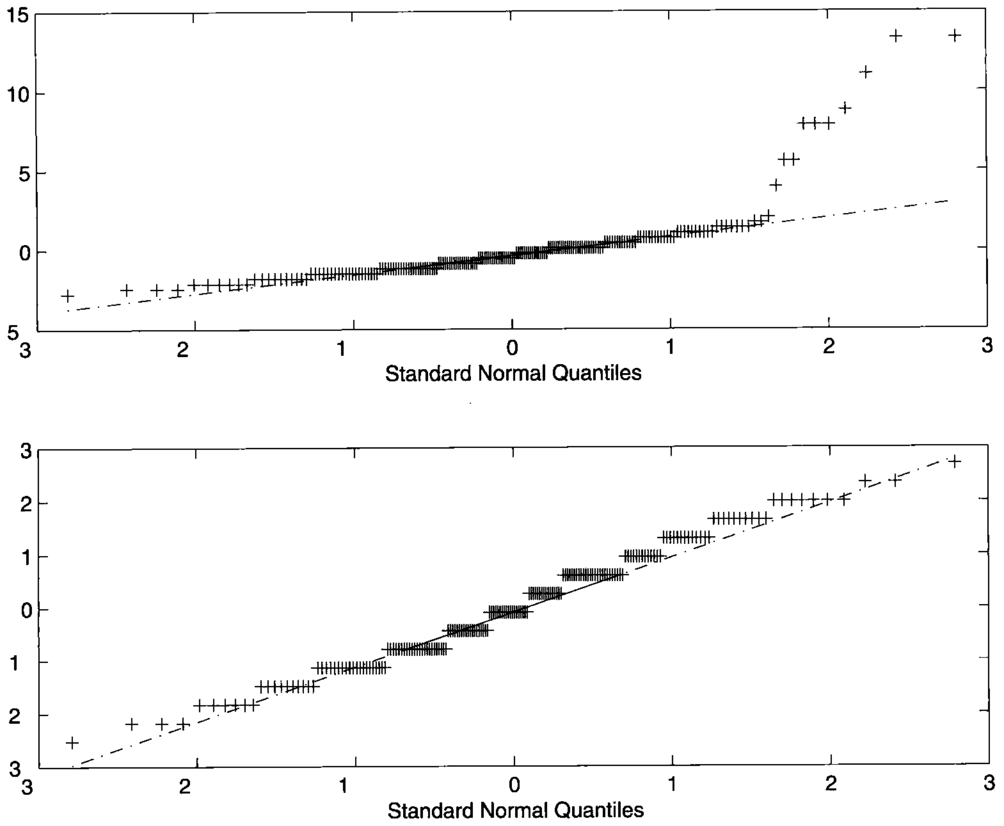

FIGURE 2. The Normal QQ Plot for the Transformed Counts Data Z in Simulation Example 1.

Equation (9) by treating the estimated values of z as observed data. The algorithm consists of two steps: the *E step* (estimating z) and the *M step* (finding the MLE using the estimates in the E step).

E Step

In the pth iteration, z_i is replaced (estimated) by its expected value $\xi_i^{(p)}$ in the full likelihood in Equation (9), given current parameter estimates $\lambda_0^{(p)}, \lambda_a^{(p)}, \omega^{(p)}, \beta^{(p)}$, where

$$P(Z_i = r) = \begin{cases} \omega^{(p)} \exp(-\lambda_a^{(p)} \tau_i^{\beta^{(p)}}) \prod_{j=1}^{k_i} v_a^{(p)}(t_{ij}) & \text{if } r = 1 \\ (1-\omega^{(p)}) \exp(-\lambda_0 \tau_i^{\beta^{(p)}}) \prod_{j=1}^{k_i} v_0^{(p)}(t_{ij}) & \text{if } r = 0. \end{cases}$$

M Step

By setting the first derivative of the full log-likelihood function from Equation (9) to zero, we generate the following estimating equations:

$$\lambda_a^{(p+1)} = \frac{\sum_{i=1}^n \xi_i^{(p)} k_i}{\sum_{i=1}^n \xi_i^{(p)} \tau_i^{\beta^{(p+1)}}},$$

$$\lambda_0^{(p+1)} = \frac{\sum_{i=1}^n (1-\xi_i^{(p)}) k_i}{\sum_{i=1}^n (1-\xi_i^{(p)}) \tau_i^{\beta^{(p+1)}}}$$

$$\frac{\sum_{i=1}^n k_i}{\beta^{(p+1)}} = \left\{ \sum_{i=1}^n (1-\xi_i^{(p)}) \lambda_0^{(p)} \tau_i^{\beta^{(p+1)}} \log(\tau_i) + \xi_i^{(p)} \lambda_a^{(p)} \tau_i^{\beta^{(p+1)}} \log(\tau_i) \right\} - \sum_{i=1}^n \sum_{j=1}^{k_i} \log(t_{ij}), \quad (11)$$

and ω is updated as $\omega^{(p+1)} = \sum_{i=1}^n \xi_i^{(p)}/n$. The E step and the M step are repeated until the parameter estimates converge to the MLEs. In this case, convergence is guaranteed by Theorem 2 in Wu (1983) because the full-data likelihood is a member of the exponential family.

For the copier data, the EM steps were repeated until the parameter estimates converged to stationary points, which can be monitored by the trace of the algorithm output. The MLEs are ($\hat{\lambda}_0$, $\hat{\lambda}_a$, $\hat{\beta}$, $\hat{\omega}$) = (0.0091, 0.0229, 0.5862, 0.1439). The result shows that the systems are experiencing reliability growth by the fact $\hat{\beta} = 0.58 < 1$; about 14.4% of the total population seems to come from a subpopulation with higher failure rate. The ξ_i's from the EM algorithm can be regarded as the posterior probability of being in the nonconforming group for system i. Based on a simple rule by classifying a system as nonconforming if $\xi_i > 0.5$ (this would obviously change if a nondegenerate risk function were used), machines 6, 16, 20 are classified as nonconforming by the fact that $\xi_6 = 0.74$, $\xi_{16} = 0.9174$, and $\xi_{20} = 0.5760$.

PLP Model Inference

Titterington (1990) has shown that inference for mixture distributions can be fraught with problems of nonidentifiability and unsolvable likelihoods. In this case, we are assuming the mixture has two components, which greatly simplifies the problem structure. For testing $H_0 : \omega = 0$ versus $H_a : \omega > 0$, the likelihood ratio

$$\Lambda = \frac{\sup_{H_0} L(\boldsymbol{\theta}; t)}{\sup_{H_a} L(\boldsymbol{\theta}; t)} \qquad (12)$$

is simple enough to compute. Under standard regularity conditions for the likelihood (see Lehmann (1997), for example), $X^2 = -2\log\Lambda$ is distributed as ${\chi_1}^2$. However, likelihood-based procedures are not guaranteed even in this case; the regularity conditions on the parameter space that satisfy requirements for MLE limit properties cannot be met. For the null hypothesis of homogeneity, the parameter space includes parameter boundary values $\omega = 0$ along with the line $\lambda_0 = \lambda_a$, corresponding to a nonidentifiable subset of the parameter space $\Theta = \{(\omega, \lambda_0, \lambda_a, \beta) \in ([0,1], (\Re^+)^3)\}$.

In place of a conventional likelihood ratio test, computational methods can be used for tests and confidence regions for unknown parameters based on resampling methods, as demonstrated in Feng and McCulloch (1996). For the hypotheses

$$H_0 : v(t) = v_0(t)$$

versus

$$H_a : v(t) = (1-\omega)v_0(t) + \omega v_a(t),$$

an approximate test is constructed by the following bootstrap likelihood-ratio procedure:

1. Compute the MLE $\hat{\boldsymbol{\theta}}_0$ of $\boldsymbol{\theta}_0 = (\lambda, \beta)$ under H_0.
2. Generate a bootstrap sample corresponding to the $\hat{v}_0(t)$, where the unknown parameters are replaced by the MLE $\hat{\boldsymbol{\theta}}_0$.
3. Compute the test statistic $X^2 = -2\log\Lambda$ corresponding to Equation (12) after finding two sets of MLEs.
4. Repeat these last two steps B times ($B > 1000$, at least) and store the B values of the test statistics ${X_1}^2, \ldots, {X_B}^2$.
5. Compute the significance of X^2 using the distribution of the B test statistics as the null distribution.

From these steps, the replicated values of $-2\log\Lambda$ formed from the successive bootstrap samples provide an assessment of the bootstrap, i.e., the null distribution of $-2\log\Lambda$. The jth order statistic in the B replications can be taken as an estimate of the $100j/B$ percentile of the null distribution. Thus, the p-value can be approximated by comparing the bootstrapped samples with the original X^2 test statistic.

The bootstrap approach can also be used to study the standard errors of the MLE for $\boldsymbol{\theta} = (\omega, \lambda_a, \lambda_0, \beta)$. A simple nonparametric bootstrap is applied here to avoid the complexity of simulating the nonhomogeneous Poisson process. We first construct B bootstrap samples, $t_1^*, t_2^*, \ldots, t_B^*$, by resampling with replacement from the n observation systems. Let $\hat{\boldsymbol{\theta}}_1^*, \ldots, \hat{\boldsymbol{\theta}}_B^*$ be the bootstrap estimates of $\boldsymbol{\theta}$ calculated from $t_1^*, \ldots, t_B^*$, respectively, using the EM algorithm. The covariance matrix of $\hat{\boldsymbol{\theta}}$ can be estimated using the sample covariance matrix of $\hat{\boldsymbol{\theta}}_1^*, \ldots, \hat{\boldsymbol{\theta}}_B^*$,

$$V = \sum_{k=1}^{B} (\hat{\boldsymbol{\theta}}_k^* - \bar{\boldsymbol{\theta}}^*)(\hat{\boldsymbol{\theta}}_k^* - \bar{\boldsymbol{\theta}}^*)^T/(B-1),$$

where $\bar{\boldsymbol{\theta}}^* = \sum_{k=1}^{B} \hat{\boldsymbol{\theta}}_k^*/B$.

Under H_0, the repair data for copy-machine failures lead to $(\hat{\lambda}, \hat{\beta}) = (0.0134, 0.5639)$ and the log likelihood ratio is calculated as $X^2 = 2.4756$. Based on $B = 2000$ bootstrap samples representing the null distribution, the p-value for the original repair data is 0.32. This lack of strong evidence is due, in part, to the small sample size of $n = 20$ for the mixture problem.

For Simulation Example 1, the histogram for model parameters using nonparametric bootstrap method is shown in Figure 3. The histograms show that all the distributions are approximately symmetric.

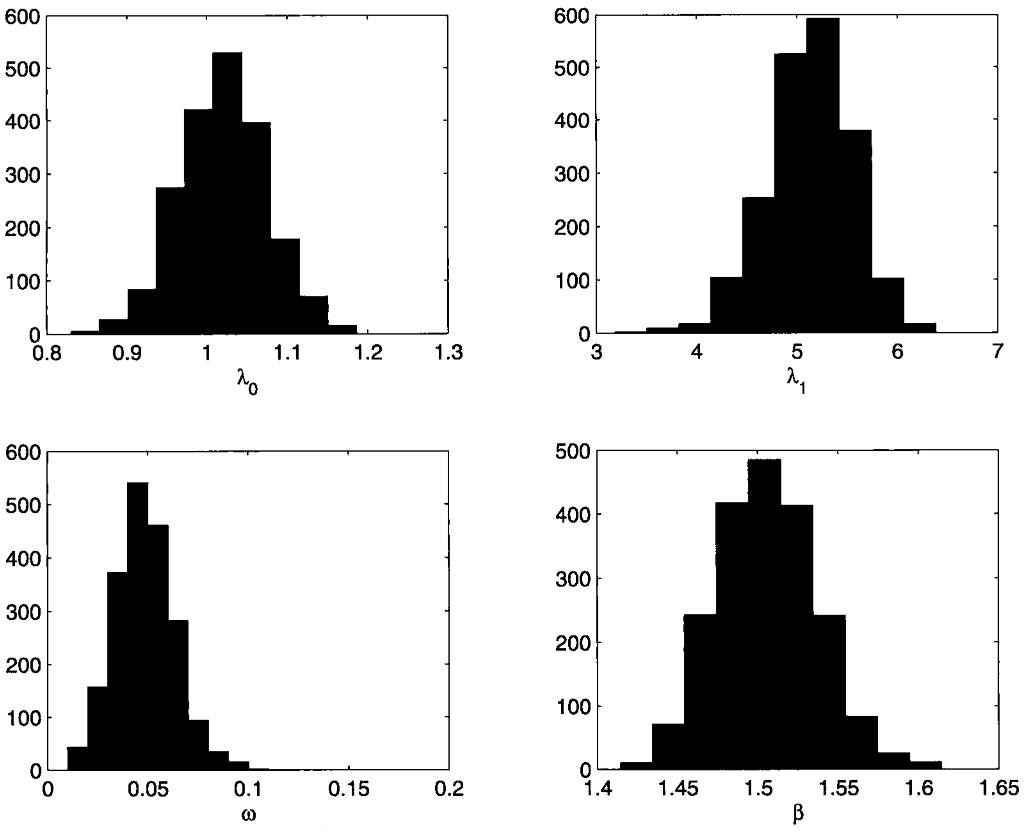

FIGURE 3. The Histograms for the Model Parameters in Mixture Power Law Processes Based on Bootstrapped Samples in Simulation Example 1.

Remark 1

The exact point estimates for both parameters as well as an exact interval estimate for the shape parameter for a single system are well studied (Finkelstein (1976)). For multiple systems with identical λ and β, the asymptotic properties for MLEs $\hat{\lambda}$ and $\hat{\beta}$ can be derived. To keep this presentation short, we consider the case where all the systems are failure truncated on the right with the same failure number m. By letting the number of systems $n \to \infty$ and the failure number $m \to \infty$, the asymptotic confidence intervals for $\hat{\lambda}$ and $\hat{\beta}$ can be obtained from the Fisher information matrix as shown in Theorem 1 below.

Remark 2

For the homogeneous population, another estimator for the shape parameter,

$$\tilde{\beta} = \frac{\sum_{i=1}^{n} k_i}{\sum_{i=1}^{n} \sum_{j=1}^{k_i} \log\left(\frac{T_i}{t_{ij}}\right)}, \qquad (13)$$

is called the *conditional* MLE; Rigdon et al. (1998) showed that conditional on system i having K_i failures, the random variable $2K\beta/\tilde{\beta}$ has an approximate χ^2 distribution with $2K$ degrees of freedom, where $K = \sum_{j=1}^{n} K_i$. The transformed random variables $U_{ij} = \log(T_i/T_{i,k_i-j+1})$ are distributed as K_i order statistics from an exponential distribution with (unknown) mean $1/\beta$. The standard estimator for the mean of U_{ij} is $\sum_{i=1}^{m} \sum_{j=1}^{K_i} U_{ij} / \sum_{i=1}^{m} K_i$, which simplifies to $1/\tilde{\beta}$.

By extending the limit results of Gaudoin et al. (2004) to multiple systems, the asymptotic normality of the MLEs can be obtained, allowing hypothesis tests and confidence regions to be constructed via the Fisher information. The proof of the theorem that follows is relegated to the Appendix.

Theorem 1

For i independent systems, let t_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, m$ be the failure times from system i, where failure times are governed by a power law process with parameter vector $\boldsymbol{\theta} = (\lambda, \beta)$. Then, under the standard regularity conditions for MLEs, as $n \to \infty$ and $m \to \infty$,

$$\sqrt{nm}(\hat{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}) \to N(0, \mathcal{I}(\bar{\boldsymbol{\theta}})^{-1}),$$

where

$$\mathcal{I}(\bar{\boldsymbol{\theta}})^{-1} = \begin{pmatrix} \lambda^2[1 + (\log \frac{m}{\lambda})]^2 & -\lambda\beta \log \frac{m}{\lambda} \\ -\lambda\beta \log \frac{m}{\lambda} & \beta^2 \end{pmatrix} \quad (14)$$

is the inverse of the Fisher information matrix.

Optimal Strategy in Warranty Decision Making

Suppose that, from the recent repair history of a group of similar systems, we know the intensity parameters for the nonconforming and conforming systems are λ_a and λ_0, respectively. Further suppose that, under the minimal repair warranty policy, failed products experience minimal repair without any cost to the consumers, but the manufacturer incurs a cost of $C_m > 0$ per repair. Let t_w be the length of the warranty coverage. Then the expected total repair costs for conforming systems and nonconforming systems are $C_m\Lambda_0(t_w)$ and $C_m\Lambda_a(t_w)$, respectively, where $\Lambda_0(t_w) = \lambda_0 t_w^\beta$ and $\Lambda_a(t_w) = \lambda_a t_w^\beta$. If the minimal repair costs for nonconforming products are high enough (compared with the fixed cost C_T of system replacement), we can lower the total repair costs by identifying and removing those nonconforming systems before t_w.

Consider the case where the products are examined after k failures, i.e., the product lifetimes are failure truncated on the right. We classify the products into two groups based on the hypothesis test $H_0: \lambda_i = \lambda_0$ vs. $H_a: \lambda_i = \lambda_a$. The expected costs due to the classification errors are given in Table 2. Denote $P(H_a \mid H_0)$ and $P(H_0 \mid H_a)$ as the Type I and Type II errors, respectively. The total expected cost function is

$$\begin{aligned} C(k) &= m(1-\omega)P_k(H_a \mid H_0) \\ &\quad \times \left\{(C_T + C_m k) - C_m\lambda_0 t_w^\beta\right\} \\ &\quad + m\omega P_k(H_0 \mid H_a) \\ &\quad \times \left\{C_m\lambda_a t_w^\beta - (C_T + C_m k)\right\}, \end{aligned} \quad (15)$$

where $0 \le k \le \lambda_a t_w^\beta - C_T/C_m$ and $\lambda_0 t_w^\beta < C_T/C_m$, because the misclassification costs will always be larger than 0.

Corresponding to the hypothesis test $H_0: \lambda_i = \lambda_0$ vs. $H_a: \lambda_i = \lambda_a$, the likelihood ratio statistic is

$$\mathbf{LR} = \frac{\exp(-\lambda_a t_{ik}^\beta)\prod_{j=1}^k \lambda_a\beta t_{ij}^{\beta-1}}{\exp(-\lambda_0 t_{ik}^\beta)\prod_{j=1}^k \lambda_0\beta t_{ij}^{\beta-1}}$$

$$= \left(\frac{\lambda_a}{\lambda_0}\right)^k \exp[(\lambda_0 - \lambda_a)t_{ik}^\beta].$$

The uniformly most powerful (UMP) test (Lehmann (1997), p. 74) is to reject H_0 if $t_{ik} < \eta_k$, where η_k is the critical value to be decided. Under H_0, t_{ik} has a generalized gamma distribution GGAM(λ, β, k) (see Ridgon and Basu (2000), p. 57), with cumulative distribution function G given as

$$G(t; \lambda, \beta, k) = \Gamma_I(\lambda t^\beta; k),$$

and where Γ_I is the incomplete gamma function defined by $\Gamma_I(v;k) = \int_0^v x^{k-1}\exp(-x)dx/\Gamma(k)$, $v > 0$. By controlling the Type I error level at α, the critical value η_k can be solved from

$$\Gamma_I(\lambda_0\eta_k^\beta; k) = \alpha.$$

Then the Type II error can be calculated as

$$\begin{aligned} P_k(H_0|H_a) &= 1 - \Gamma_I(\lambda_a\eta_k^\beta; k) \\ &= 1 - \Gamma_I[\Gamma_I^{-1}(\alpha;k)\lambda_a/\lambda_0], \end{aligned} \quad (16)$$

where $\Gamma_I^{-1}(\,\cdot\,;k)$ is the inverse function of $\Gamma_I(\,\cdot\,;k)$. By plugging in $P_k(H_0|H_a)$ from Equation (16) into Equation (15), we know the minimum expected cost must be contained in the interval $k \in [1, \lambda_a t_w^\beta - C_T/C_m]$.

Observation 1

If ω is small such that the nonconforming products do not affect the total costs in Equation (15) as much as conforming products, $C(k)$ is an increasing function in k, and the manufacturer benefits from earlier testing.

Observation 2

Figure 4 shows the Type II error as a function of the ratio λ_a/λ_0 under $\alpha = 0.05$. If the ratio $\lambda_a/\lambda_0 > 5$, we can see that the Type II error approaches 0 quickly as k increases. When λ_a/λ_0 is large, nonconforming products are more easily detected even without a large failure number k.

TABLE 2. The Cost Functions for Misclassifications

Probability	Cost function
$P(H_0 \mid H_0)$	$C_m\lambda_0 t_w^\beta$
$P(H_a \mid H_0)$	$C_m k + C_T$
$P(H_0 \mid H_a)$	$C_m\lambda_a t_w^\beta$
$P(H_a \mid H_a)$	$C_m k + C_T$

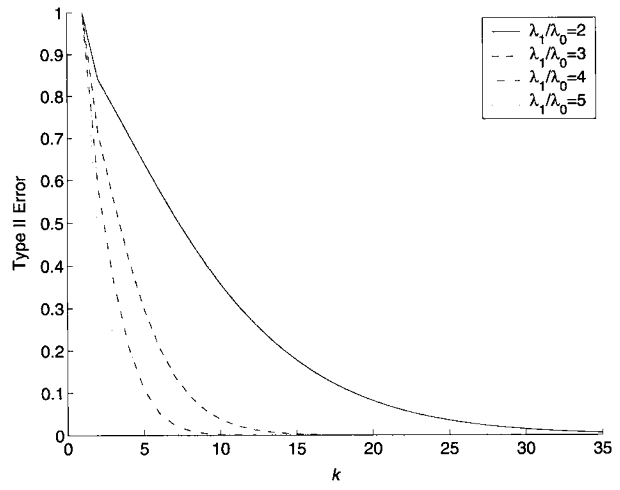

FIGURE 4. Type II Error from Equation (16) Using Ratios of Different Intensity Parameters.

Simulation Example 2

We illustrate this optimal decision process through the following simulation. The total warranty coverage time is set as $t_w = 4$ years, and we use four different values of β (i.e., $\beta \in \{0.5, 1, 1.5, 2\}$) to illustrate reliability varying from reliability growth to deterioration.

The proportion of nonconforming systems ω is set to one of four different values ($\omega \in \{0.001, 0.01, 0.05, 0.1\}$) to compare different scenarios of population contamination. Without loss of generality, λ_0 is chosen to be 2, and corresponding to this, λ_a is set to be three, five, or ten times the value of λ_0, i.e., $\lambda_a \in \{3\lambda_0, 5\lambda_0, 10\lambda_0\}$. Finally, C_T/C_m is assumed to be equal to $(\lambda_a + \lambda_0)t_w^{\beta}/2$. The optimal values of k under each of the different cases are shown in Table 3, which supports observations 1 and 2 above. For example, with $\lambda_a = 3\lambda_0$, $\beta = 2$, and $\omega = 0.05$, the expected cost is minimized by choosing $k = 8$. That is, the optimal strategy is to test each system until eight failures occur before deciding whether or not the system is conforming or nonconforming.

TABLE 3. The Optimal k Under Different Model Parameters in a Simulated Process

ω	$\beta = 0.5$	$\beta = 1$	$\beta = 1.5$	$\beta = 2$
		$\lambda_a/\lambda_0 = 3$		
0.001	1	1	1	1
0.01	1	1	1	3
0.05	2	4	6	8
0.1	4	6	8	10
		$\lambda_a/\lambda_0 = 5$		
0.001	1	1	1	1
0.01	1	2	3	4
0.05	3	4	5	6
0.1	4	5	6	6
		$\lambda_a/\lambda_0 = 10$		
0.001	1	1	1	2
0.01	2	2	3	3
0.05	3	3	3	4
0.1	3	3	4	4

Conclusion

This paper studies the modeling of heterogenous systems governed by a minimal repair process. An exploratory study and graphical methods are used to detect heterogeneity of the power law processes for 20 copy machines based on repeated failure-time data. Bootstrap methods are used to calibrate the estimation uncertainty as well as likelihood ratio test statistics.

When considering a model for conforming and noncomforming systems, the two-point mixture model makes intuitive sense and is easily interpreted. Furthermore, it lends itself to a natural formula for classifying products as nonconforming or conforming. However, discrete mixtures are difficult to fit, especially with small samples. Alternatively, the *continuous* mixture model generated with a Gamma mixing distribution for λ (Englehardt and Bain (1987)) will fit the copy-machine failure data, but the estimated mixing parameters from the Gamma distribution are poorly fit, especially the shape parameter. This is due, in part, to the small sample size.

Finally, an optimal decision based on estimated values is derived to minimize warranty cost. The decision process is aided by "missing-data" estimates in the EM Algorithm. Future study can consider more complex warranties based on intricate risk functions. Our asymptotic results are based on a simple system of minimal repair with failure truncation on the right, and confidence statements for the power law process parameters can be constructed from the Fisher information matrix of Theorem 1.

Appendix
Proof of Theorem 1

The asymptotic normality of the parameter estimates for a single system is demonstrated in Gaudoin et al. (2004). To shorten this presentation, we only illustrate the derivation of the asymptotic covariance through the information matrix. The likelihood for the repair times can be expressed as

$$L(\beta, \lambda; t) \propto \prod_{i=1}^{n} \left\{ \exp(-\lambda t_{im}^{\beta}) \prod_{j=1}^{m} \lambda\beta t_{ij}^{\beta-1} \right\},$$

and the corresponding Fisher Information matrix is obtained as

$$\mathcal{I} = \begin{pmatrix} -\mathrm{E}\left(\frac{\partial^2 \log L}{\partial \lambda^2}\right) & -\mathrm{E}\left(\frac{\partial^2 \log L}{\partial \lambda \partial \beta}\right) \\ -\mathrm{E}\left(\frac{\partial^2 \log L}{\partial \lambda \partial \beta}\right) & -\mathrm{E}\left(\frac{\partial^2 \log L}{\partial \beta^2}\right). \end{pmatrix}$$

This simplifies to

$$\mathcal{I} = \begin{pmatrix} \frac{nm}{\lambda^2} & \sum_{i=1}^{n} A_{m,\beta}(i) \\ \sum_{i=1}^{n} A_{m,\beta}(i) & \frac{nm}{\beta^2} \sum_{i=1}^{n} (A_{m,\beta}(i))^2), \end{pmatrix}$$

where

$$A_{m,\beta}(i) = \mathrm{E}(T_{i,m}^{\beta} \log T_{i,m}).$$

Using results derived in Crow (1974) and Gaudoin (2004), we have

$$\sum_{i=1}^{n} \mathrm{E}(T_{i,m}^{\beta} \log T_{i,m}) = \frac{nm}{\lambda\beta}[\lambda(m+1) - \log\lambda]$$

and

$$\sum_{i=1}^{n} \mathrm{E}(T_{i,m}^{\beta} \log^2 T_{i,m}) = \frac{nm}{\lambda\beta^2}[\lambda^{(1)}(m+1) + (\lambda(m+1) - \log\lambda)^2],$$

where $\phi(z) = \partial \log\Gamma(z)/\partial z$ is the digamma function and $\phi^{(1)}(z) = \partial\phi(z)/\partial z$ is the polygamma function of order 1. By the equivalency of $\phi(m)$ with $\log m$ and $\phi^{(1)}(m)$ with $1/m$, the information matrix can be inverted to $\mathcal{I}^{-1}$ in Equation (14) from the theorem.

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