The Backward Shift on the Space of Cauchy Transforms

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THE BACKWARD SHIFT ON THE SPACE
OF CAUCHY TRANSFORMS

JOSEPH A. CIMA, ALEC MATHESON, AND WILLIAM T. ROSS

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Abstract. This note examines the subspaces of the space of Cauchy transforms of measures on the unit circle that are invariant under the backward shift operator \( f \rightarrow z^{-1}(f - f(0)) \). We examine this question when the space of Cauchy transforms is endowed with both the norm and weak* topologies.

1. Introduction and preliminaries

In this note, we will examine the invariant subspaces of the backward shift operator

\[
(Bf)(z) = \frac{f(z) - f(0)}{z}
\]

on the space of Cauchy transforms \( K \) consisting of analytic functions on the open unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) that take the form

\[
(K\mu)(z) := \frac{d\mu(\zeta)}{1 - \zeta z}
\]

Here \( \mu \in M \), the space of finite Borel measures on the unit circle \( T = \{ z \in \mathbb{C} : |z| = 1 \} \).

By an “invariant subspace” of \( K \) we will mean a closed linear manifold \( M \subseteq K \) for which \( bM \subseteq M \). In using the word “closed”, there are two topologies on \( K \) to consider here. The first is the norm topology. For \( f \in K \), let

\[
M_f := \{ v \in M : f - Kv \}
\]

be the set of “representing measures” for \( f \). Define the norm of an element \( f \in K \) by

\[
\| f \| := \inf \{ 1 \nu 1 : v \in M_f \},
\]

where \( 1 \nu 1 \) denotes the total variation norm of the measure \( v \). The notation \((K, 1 \cdot 1)\) will denote the space \( K \) endowed with the above norm topology. It is well known that \((K, 1 \cdot 1)\) is isometrically isomorphic to the quotient space \( M/H^0 \), and is a non-separable Banach space. Here \( H^0 \) is the usual Hardy space of the disk \([0]\) and \( H^0 \) are the functions in \( H^1 \) that vanish at the origin. \( H^0 \) is regarded as a subspace of \( M \) in the natural way as \( \{ f dm : f \in H^1 \} \), where \( dm = |d\zeta|/2\pi \) is normalized Lebesgue measure on the circle. The second topology on \( K \) is the weak* topology.
that arises by identifying the dual space of the disk algebra \( A \) (analytic functions on \( D \) that have continuous extensions to \( D^* \)) with \( K \) via the pairing
\[
(f; K\mu) = \int f d\mu, \quad f \in A, \mu \in M.
\]

By the F. and M. Riesz theorem [9, p. 41], if \( \mu_1, \mu_2 \in M_{K\mu} \), then \( d\mu_1 - d\mu_2 = \mu \), where \( h \in H^1_\mathcal{E} \). Thus the above pairing is independent of the representing measure \( \mu \). We will use the notation \((K, \ast)\) to denote the space \( K \) endowed with the weak* topology. One can show that \((K, \ast)\) is separable. Furthermore, every weak* closed subspace of \( K \) is norm closed. See [4], [5], and [6] for a review of these basic facts about \( K \). In this paper, we examine the \( B \)-invariant subspaces of \((K, \ast)\) and \((K, 1 \cdot 1)\).

To put our results in perspective, we mention some known results about the \( B \)-invariant subspaces for other spaces of analytic functions. For example, by Beurling's theorem [9, p. 114], the \( B \)-invariant subspaces of the classical Hardy space \( H^2 \) all take the form \((\mathbb{H} \mathcal{E})^\perp\), where \( \mathbb{H} \) is an inner function. Moreover [8] (see also [6]), \( f \) belongs to \((\mathbb{H} \mathcal{E})^\perp\) if and only if there is a function \( G_t \in N^+(D_\mathcal{E}) \) that vanishes at infinity such that
\[
(1.2) \quad \lim_{r \to 1^-} f(r\zeta) = \lim_{r \to 1^-} G_t(\zeta/r)
\]
for \( m \)-almost every \( \zeta \in \mathbb{T} \). Here \( D_\mathcal{E} := \mathbb{H} \mathcal{E} \mathcal{D}^- \) and \( G_t \in N^+(D_\mathcal{E}) \) means \( G_t(1/\zeta) \in N^+ \) (the Smirnov class of \( D \) [9, p. 25]). The function \( G_t \) is called a "pseudocontinuation"\(^1\) of \( f \). If
\[
\sigma(\mathcal{E}) := \{ \zeta \in \mathbb{D}^- : \lim_{\lambda \to \zeta} |\mathcal{E}(\lambda)| = 0 \},
\]

then, by basic properties of inner functions [11, pp. 68 and 69], \( \mathcal{E} \) has an analytic continuation to \( \mathcal{E} \mathcal{F} \mathcal{P} \sigma(\mathcal{E})^* \), where \( \sigma(\mathcal{E})^* := \{ \zeta \in \mathcal{E} : 1/\zeta \in \sigma(\mathcal{E}) \} \). In fact, every \( f \in (\mathbb{H} \mathcal{E})^\perp \) has an analytic continuation to \( \mathcal{E} \mathcal{F} \mathcal{P} \sigma(\mathcal{E})^* \) [8].

For the Bergman space \( L^\alpha_\mathcal{E} \) (analytic functions \( f \) on \( D \) such that \( f \in L^\alpha(dx, dy) \)) a theorem of Richter and Sundberg [14] says that every \( B \)-invariant subspace takes the form \( M_g := \{ f \in L^\alpha_\mathcal{E} : f \perp \mathcal{E}_n \mathcal{D} \forall n \in N \cup \{0\} \} \) for some \( g \) in the Dirichlet space (i.e., \( g^2 \in L^2_\mathcal{E} \)). Here we equate the dual of \( L^\alpha_\mathcal{E} \) with the Dirichlet space via the "Cauchy" dual pairing
\[
\lim_{r \to 1^-} f(r\zeta) g(r\zeta) \ dm(\zeta).
\]

Furthermore, (i) \( gM_g \subseteq H^p \) for all \( 0 < p < 1 \), (ii) for every \( f \in M_g \), the meromorphic function \( f/\partial g \) (where \( \partial g \) is the inner factor of \( g \)) has a pseudocontinuation as in (1.2), (iii) every \( f \in M_g \) has an analytic continuation to \( \mathcal{E} \mathcal{F} \mathcal{P} \sigma(g)^* \). Moreover [2], if \( g \) is "sufficiently smooth", then \( gM_g \subseteq H^1 \) and \( f \in L^2 \) belongs to \( M_g \) if and only if (a) \( f_g \in H^1 \) and (b) \( f/\partial g \) has pseudocontinuation as in (1.2). For certain \( L^p \) Bergman spaces, the function \( g \) can always be chosen to be "sufficiently smooth"; so in this case we have a complete characterization of the \( B \)-invariant subspaces. Our purpose here is to get similar-looking results for the space \((K, \ast)\) (which can be gleaned from results of Korenblum [13]) and to examine the more difficult problem of characterizing the \( B \)-invariant subspaces of \((K, 1 \cdot 1)\).

---

\(^1\)If \( h \) is meromorphic on \( D \) and \( H \) is meromorphic on \( D_\mathcal{E} \) and the nontangential boundary values of \( h \) and \( H \) exist and are equal \( m \)-almost everywhere, then \( h \) and \( H \) are "pseudocontinuations" of each other. See [15] for more details.
2. The main results

For a $B$-invariant subspace $M$ of $(K, \ast)$ let

$$M_\perp = \{f \in A : (f, K\mu) = 0 \text{ for all } K\mu \in M\}$$

be the pre-annihilator of $M$. $M_\perp$ is a norm closed subspace of the disk algebra $A$.

A straightforward calculation shows that

$$\int A \ni f \mapsto \lim_{r \to 1^-} \int_{\mathbb{T}} f(\zeta)(K\mu)(r\zeta)^n dm(\zeta) = \lim_{r \to 1^- \atop n \to \infty} \hat{\mu}(r^n)$$

and $(f, BK\mu) = \langle f, K(d\mu) \rangle = \langle zf, K\mu \rangle$. Thus $zM_\perp \subseteq M_\perp$ since $BM \subseteq M$.

Since $A$ is a Banach algebra and polynomials are dense in $A$ [11, p. 17], $M_\perp$ is an ideal of $A$. A theorem of Rudin [16] (see also [11, p. 85]) says the following.

**Theorem 2.2** (Rudin). Let $I$ be a norm closed ideal of the disk algebra $A$. Then there is a closed set $E \subseteq \mathbb{T}$ of Lebesgue measure zero and an inner function $\phi$ with $\sigma(\phi) \cap T \subseteq E$ such that

$$I = I(\phi, E) := \{f \in A : f/\phi \in H^\infty, f \upharpoonright E = 0\}.$$

Furthermore, given a set $E \subseteq \mathbb{T}$ of Lebesgue measure zero and an inner function $\phi$ with $\sigma(\phi) \cap T \subseteq E$, there is an outer function $F \in A$ whose zero set is equal to $E$ and such that $\phi := \phi F$ generates $I(\phi, E)$ in the sense that the smallest norm closed ideal of $A$ containing $\phi$ is equal to $I(\phi, E)$.

To describe $M$, we need (via the Hahn-Banach theorem) to describe the set

$$(M_\perp)^\perp = I(\phi, E)^\perp := \{f \in K : (h, f) = 0 \text{ for all } h \in I(\phi, E)\}.$$

or equivalently, the set $\{f \in K : (\phi^n f, h) = 0 \forall n \in \mathbb{N} \cup \{0\}\}$. Korenblum [13] proved the following.

**Theorem 2.3** (Korenblum). If $K\mu \perp I(\phi, E)$, then $K\mu$ has an analytic continuation to the set $\mathbb{D} \cap \sigma(\phi)^* \cup E$.

In the process of proving our main theorem (Theorem 2.5), we will give an alternate proof of Korenblum’s theorem. Any measure $\mu \in M$ can be decomposed uniquely as

$$d\mu = \phi dm + d\mu_\perp,$$

where $\phi \in L^1(m)$ and $\mu_\perp \perp m$. Our main theorem describes $I(\phi, E)^\perp$.

**Theorem 2.5.** For $\mu \in M$, $K\mu \perp I(\phi, E)$ if and only if

1. the support of $\mu_\perp$ is contained in $E$;
2. $K\mu/\phi$ has an analytic continuation across $\mathbb{T} \setminus E$ to a function $F \in N^\infty(D_\phi)$ with $F(\infty) = 0$.

By the F. and M. Riesz theorem, every measure $\nu \in M$ ($f \in K$) has the same singular part. Thus in condition (1), there is only one singular part to consider. In $H^2$, the $B$-invariant subspace $(\phi H^2)^\perp$ is singly generated by the vector $f = B\phi$. This next corollary is the analogue of this for $(K, \ast)$.

**Corollary 2.6.** $I(\phi, E)^\perp = \{B^nf : n \in \mathbb{N} \cup \{0\}\}$, where $f = B(K\mu)$ for $d\mu = \phi dm + d\mu_\perp$ and $\mu_\perp \perp m$ with support equal to $E$. 


Here $V$ is the closed linear span in $(K, \ast)$. This next corollary mimics what happens in the Bergman space setting. By a classical result of Smirnov [9, p. 39], $K \subseteq H^p$ for all $0 < p < 1$, and so if our $B$-invariant subspace $M \subseteq K$ has the property that $M_\perp$ is generated by $f$, i.e., $M_\perp$ is the closed linear span (in $A$) of $z^n f (n \in N \cup \{0\})$, then certainly $fM \subseteq H^p$ for all $0 < p < 1$. If $f$ is sufficiently smooth, we get the stronger condition $fM \subseteq H^1$ and even a bit more.

**Theorem 2.7.** Suppose $f \in A$ with $f^t \in H^\infty$. Let $E = f^{-1}(\{0\}) \cap T$, and let $\theta_t$ be the inner factor of $f$. Then $K_\mu \perp z^n \theta_t f$ for all $n \in N \cup \{0\}$ if and only if

1. $fK_\mu \in H^1$;
2. $K_\mu \theta_t$ has an analytic continuation across $T \setminus E$ to a function $F \in N^1(D_0)$ with $F(\infty) = 0$.

If $f \in A$ with $f^t \in H^\infty$, then the boundary zero set $E$ of $f$ satisfies the so-called Carleson condition: If $(l_n)$ is the sequence of arcs contiguous to $E$ on the circle, then $\int_{l_n} |l_n| \log |l_n| > -\infty$. Thus, by Theorem 2.2, not every $B$-invariant subspace of $(K, \ast)$ is singly generated by such an $f$.

Comments about the $B$-invariant subspaces of $(K, 1 \cdot 1)$ appear at the end of this note.

3. The proofs

**Proposition 3.1.** Suppose $\theta F$ is a generator for $l(\theta, E)$ and $d\mu = \phi d\mu + d\mu_s$ as in (2.4). Then $K_\mu \perp z^n \theta F$ for all $n \in N \cup \{0\}$ if and only if $\phi \in \theta H^\infty_0$ and $\mu_s$ is supported in $E$.

**Proof.** Suppose $K_\mu \perp z^n \theta F$ for all $n \in N \cup \{0\}$. Then, by (2.1),

$$
\int_T r z^n \theta F (\phi d\mu + d\mu_s) = 0 \quad \text{for all } n \in N \cup \{0\}.
$$

From the F. and M. Riesz theorem, $\theta F d\mu_s$ is the zero measure (and so $\mu_s$ is supported in $E$) and $\theta F \phi \in \Pi$. However, $\phi \theta = \theta F \in \Pi$ and has $L^1(m)$ boundary values, and so $\phi \theta \in H^\infty_0$ [9, p. 28]. The converse is obvious.

**Proof of Theorem 2.5.** We start by proving a somewhat weaker result: $K_\mu \perp l(E, \theta)$ if and only if $\mu_s$ is supported in $E$ and $K_\mu \theta$ has a pseudocontinuation across $T$ belonging to $N^1(D_0)$ and vanishing at infinity. Indeed, suppose $K_\mu \perp l(E, \theta)$. By Proposition 3.1 we can assume $\mu$ takes the form

$$
d\mu = \phi d\mu + d\mu_s, \quad \phi \theta = k \in H^\infty_0 \quad \text{supp}(\mu_s) \subseteq E.
$$

Since $k \in H^\infty_0$, then $k(1/z)$ belongs to $H^1(D_0)$ and vanishes at infinity. The inner function $\theta$ is defined on $D_0$ by $\theta(z) = 1/k(1/z)$. The function

$$
\psi(z) := \frac{d\mu(z)}{1 - \zeta z}, \quad z \in D_0
$$

belongs to $H^p(D_0)$ for all $0 < p < 1$ [9, p. 39] and so the function

$$
T_{\mu, \theta}(z) := k(1/z) + \frac{\psi(z)}{\theta(z)}, \quad z \in D_0
$$

is a function in $l(E, \theta)$.
belongs to \( N'(D_\theta) \) and vanishes at infinity. By Fatou's jump theorem\(^2\), the boundary function for \( T_{\mu,\theta} \) is

\[
\frac{\varphi}{\mathcal{G}}(\zeta) + \frac{(K\mu)(\zeta) - \varphi(\zeta)}{\mathcal{G}(\zeta)} = \frac{K\mu}{\mathcal{G}}(\zeta)
\]

for \( m \)-almost every \( \zeta \in T \). Thus \( T_{\mu,\theta} \) is the pseudocontinuation of \( K\mu/\mathcal{G} \) of the desired type.

Conversely, suppose \( d\mu = \varphi dm + d\mu_\phi \), where \( \varphi \in L^1(m) \) and \( \mu_\phi \) is supported in \( E \), and \( K\mu/\mathcal{G} \) has a pseudocontinuation \( G \in N'(D_\theta) \) with \( G(\infty) = 0 \). Then, by Fatou's jump theorem,

\[
G(\zeta) = \lim_{r \to 1^-} \frac{K\mu}{\mathcal{G}}(r\zeta) = \frac{\varphi(\zeta) + \phi(\zeta)}{\mathcal{G}(\zeta)}.
\]

Assuming for the moment that \( \mathcal{G}(0)/= 0 \), we conclude that \( G - e/\mathcal{G} \in N'(D_\theta) \) and vanishes at infinity. Then \( \varphi/\mathcal{G} \) is the boundary function of a function from \( N'(D_\theta) \) that vanishes at infinity. But since \( \varphi/\mathcal{G} \in L^1(m) \), then \( \varphi/\mathcal{G} \in H^1 \). If \( \mathcal{G}(0) = 0 \), then use the same argument with \( \mathcal{G} \) replaced by \( \mathcal{G}z^0 \) and \( G \) replaced by \( Gz^0 \) for some positive integer \( n \). Now apply Proposition 3.1.

Now we need to show that \( K\mu \) has an analytic continuation to \( \mathcal{G}(\mathcal{G}^*) \cup E \).

As mentioned earlier, this was originally shown by Korenblum in [13]. Indeed, if \( W \subseteq \mathcal{G}(\mathcal{G}^*) \cup E \) is an open set containing an arc of the circle, then \( T_{\mu,\theta} \) (as defined in (3.3)) is analytic on \( W \cap D_\theta \) and by standard estimates,

\[
|T_{\mu,\theta}(\lambda)| \leq C1\mu \frac{1}{|\lambda| - 1}, \quad \lambda \in W \cap D_\theta.
\]

Since \( K\mu \perp I(\mathcal{G}, E) \), we can apply Proposition 3.1 to conclude that \( \mu \) takes the form

\[
d\mu = \varphi dm + d\mu_\phi,
\]

where \( \varphi = \mathcal{G}(h \in H^1) \) and \( \mu_\phi \) is supported in \( E \).

Next, let \( (h_n) \) be a sequence of polynomials in \( H_0 \) that approximates \( h \) in norm and set

\[
d\mu_n := \mathcal{G}h_n dm + d\mu_\phi.
\]

Notice that \( 1\mu_n \) is uniformly bounded in \( n \). By Proposition 3.1, \( K\mu_n \perp I(\mathcal{G}, E) \) and the corresponding pseudocontinuation of \( K\mu_n/\mathcal{G} \) is

\[
T_{\mu_n,\theta}(z) = \mathcal{G}(1) + \frac{r}{\mathcal{G}(z)} \mathcal{G}(h\mathcal{G}(z)) \frac{1}{1 - \overline{\zeta} z} dm(z) + \frac{1}{\mathcal{G}(z)} d\mu_\phi(\zeta).
\]

Since the functions \( \mathcal{G}h_n \) are bounded on \( T \), then \( K\mu_n/\mathcal{G} \) and \( T_{\mu_n,\theta} \) are \( H^1 \) functions on \( W \cap D \) and \( W \cap D_\theta \) (respectively) [9, p. 41]. (Note that \( \mathcal{G} \) has an analytic continuation across \( W \cap T \) as does \( \mu_\phi \) since this \( W \cap T \) avoids the support of \( \mu_\phi \).) Moreover, by what was said earlier, they have equal boundary values almost everywhere on \( W \cap T \). By a standard Morera's theorem argument [10, p. 95], these two functions are analytic continuations of each other across \( W \cap T \).

\(^2\)Fatou's jump theorem: \( \lim_{r \to 1^-} -\mathcal{G}(r^*) = \lim_{r \to 1^-} \mathcal{G}(r) = \lim_{r \to 1^-} \mathcal{G}(r) = \lim_{r \to 1^-} \mathcal{G}(r) = \mathcal{G}(r) = \mathcal{G}(r) = \mathcal{G}(r) \) \( m \)-almost everywhere [9, p. 4].
Finally,
\[
\begin{align*}
\frac{\mathcal{K}_\mu}{\vartheta} \left( \lambda \right) & \leq C \mu_{\vartheta 1} \frac{1}{1-|\lambda|} \frac{1}{C} \mathcal{C} \left( \lambda \right), \quad \lambda \in W \cap \mathbb{D}, \\
|T_{\mu_\vartheta 1}(\lambda)| & \leq C \mu_{\vartheta 1} \frac{1}{|\lambda|} \frac{1}{C} \mathcal{C} \left( \lambda \right), \quad \lambda \in W \cap \mathbb{D}. 
\end{align*}
\]

By a theorem of Beurling [7] (see also [15, p. 95]), these functions form a normal family on \( W \) and so a subsequence (for which we will use the same index) converges to an analytic function on \( W \). But since \( K_{\mu_\vartheta} \) converges pointwise to \( K_\mu/\vartheta \), then \( K_\mu \), and hence \( K_\mu \), will have an analytic continuation to \( W \).

**Proof of Corollary 2.6.** If \( \vartheta F \) is a generator of the ideal \( I(\vartheta, E) \), then by our “Cauchy pairing” in (2.1), it is routine to show that
\[
(z^m \vartheta F, B^n f) = \vartheta F \xi^{m, n}(\vartheta dm + d\vartheta) = 0 \quad \forall m, n \in \mathbb{N} \cup \{0\}.
\]

Thus
\[
\{B^n f : n = 0, 1, 2, \ldots \} \subseteq I(\vartheta, E)^\perp.
\]

If \( g \in A \) satisfies \( \langle g, B^nf \rangle = 0 \) for all \( n \), one can use the F. and M. Riesz theorem to show that \( g/\vartheta \in H^1 \) and \( g \) is zero on the support of \( \mu_\vartheta \) (which equals \( E \)). Thus \( g \in I(\vartheta, E) \). An application of the Hahn-Banach theorem completes the proof.

The proof of Theorem 2.7 requires a few preliminaries. Notice that \( K_\mu \in L^1(dA) \), where \( dA \) is the area measure on \( \mathbb{D} \). This follows from Fubini’s theorem and the fact that the integral
\[
\frac{1}{\mathcal{D}} \int_{\mathbb{A}} \frac{1}{|e^{i\theta} - z|} \, dA(z)
\]
is uniformly bounded in \( \theta \).

For a Cauchy transform \( K_\mu \), consider the function
\[
(K_\mu)(\lambda)(z) = \frac{1}{\mathcal{D}} \int_{\mathbb{A}} \frac{1}{z - \lambda} \, dA(z), \quad \lambda \in \mathbb{D}.
\]

Since \( K_\mu \in L^1(dA) \) and is analytic on \( \mathbb{D} \), it is not difficult to show, using the fact that \( (z - \lambda)^{-1} \in L^1(dA) \) for each fixed \( \lambda \in \mathbb{D} \), that the above integral exists for every \( \lambda \in \mathbb{D} \). Moreover, the dominated convergence theorem says that the above function is continuous on \( \mathbb{D} \).

**Proposition 3.4.** For \( \mu \in M \),
\[
\sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |(K_\mu)(\lambda)(z)| \, dA(z) \, d\theta < \infty.
\]

**Proof.** For fixed \( 0 < r < 1 \),
\[
\begin{align*}
\frac{1}{2\pi} \int_0^{2\pi} & \left( |K_\mu(z)| \right) \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1}{|z - r e^{i\theta}|} \right) \, dA(z) \, d\theta \\
\frac{1}{2\pi} \int_0^{2\pi} & \left( \frac{1}{|z - r e^{i\theta}|} \right) \, dA(z) \, d\theta \\
\frac{1}{2\pi} \int_0^{2\pi} & \left( \frac{1}{|z - re^{i\theta}|} \right) \, dA(z) \, d\theta.
\end{align*}
\]

Use the standard inequality
\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|e^{i\theta} - a|} \leq C \log \left( \frac{1}{1 - |a|} \right), \quad |a| < 1.
\]
to get
\[ \int_0^{2\pi} \frac{d\theta}{|se^{\theta} - re^{\theta}|} \leq \begin{cases} \frac{1}{r} \log \left( \frac{1}{1 - s/r} \right) & \text{for } s < r, \\ \frac{1}{r} \log \left( \frac{1}{\frac{1}{1 - r/s}} \right) & \text{for } s > r. \end{cases} \]

and
\[ \int_0^{2\pi} \frac{dt}{|1 - se^{i\theta}|} C \log \left( \frac{1}{1 - s} \right). \]

Combine the above two estimates along with Fubini's theorem to show the desired integral is bounded above by
\[ \frac{C}{r} \int_0^r \log \left( \frac{1}{1 - s/r} \right) ds + \int_r^1 \log \left( \frac{1}{1 - s} \right) ds. \]

Standard estimates now show that this quantity is bounded uniformly for \( r \) close to 1.

**Proof of Theorem 2.7.** Suppose \( f \in A \) with \( f^t \in H^\infty \) and \( K\mu \perp z^nf \) for all \( n \in \mathbb{N} \cup \{0\} \). Theorem 2.5 yields condition (2). Using a power series argument, one can show that
\[ (f, K\mu) = \lim_{r \to 1} \int_D \frac{r^n}{(1 - z^r)} dm_2(z), \]

where \( dm_2 = dA/\pi \). Since \( (zf)^n \) is a bounded function, we can use the fact that \( K\mu \in L^1(dA) \), to rewrite this as
\[ \int_D (K\mu)(z)(zf)^n(z) dm_2(z). \]

For fixed \( \lambda \in D \), the function
\[ K\mu - (K\mu)(\lambda) \]

can be written as \( K\mu - \lambda \), where \( dm_\lambda = (1 - \zeta) d\mu \). By Proposition 3.1, \( K\mu - \lambda \) annihilates the ideal generated by \( f = 0 \). Thus, by what was said above,
\[ \int_D \frac{(K\mu)(z) - (K\mu)(\lambda)}{z - \lambda} \frac{(zf)^n(z)}{dm_2(z)} = 0, \quad \lambda \in D. \]

Another power series computation yields
\[ \int_D \frac{(zf)^n(z)}{z - \lambda} dm_2(z) = -(zf)^n(\lambda), \]

and so from (3.5),
\[ -(zf)^n(\lambda)(K\mu)(\lambda) = \int_D \frac{(K\mu)(z)}{z - \lambda} (zf)^n(z) dm_2(z). \]

Now use Proposition 3.4 along with the assumption that \( (zf)^n \) is bounded to show that the integrals
\[ \int_0^{2\pi} \left| f(r e^{i\theta})(K\mu)(r e^{i\theta}) \right| d\theta \]

are uniformly bounded in \( 0 < r < 1 \), that is to say, \( fK\mu \in H^1 \).

\[ \text{3See, for example, the argument used to prove Lemma 2.5 in [3].} \]
Conversely, suppose conditions (1) and (2) are satisfied. Since \( \partial_t K\mu \) and \( \bar{F}_t \) (where \( F_t \) is the outer factor of \( f \)) are the boundary values of functions from \( N^+ (D_\circ) \), then \( \bar{f} K\mu \) is also the boundary function of a \( N^+ (D_\circ) \) function that vanishes at infinity. But since \( f K\mu \in L^1(m) \), then \( \bar{f} K\mu \in \bar{H}_0^1 \). Thus

\[
(K\mu)(\zeta)\bar{\gamma}(\zeta) dm(\zeta) = 0 \quad \text{for all} \quad n \in N \cup \{0\}.
\]

Finally, using the notation \( g_r(z) := g(rz) \),

\[
(K\mu)\bar{f}_r - K\mu \bar{f} = \lim_{r \to 1^-} \frac{(K\mu \phi_r - K\mu \phi)}{f_r} \frac{f_r}{f} \frac{\bar{f}_r}{\bar{f}} \frac{\bar{f}}{f}
\]

which goes to zero in the \( L^1(m) \) norm as \( r \to 1^- \). Thus for any \( n \in N \cup \{0\} \),

\[
\int \frac{(K\mu)(\zeta)\bar{\gamma}(\zeta)}{f_r} dm(\zeta)
\]

is bounded. Hence, \( \int \bar{f} dm = 0 \) for all \( n \in N \cup \{0\} \).

### 4. The norm topology

Recall that \( (K,1 \cdot 1) \) is a nonseparable space, and so a characterization of the \( B \)-invariant subspaces is out of reach. In this final section, we will make a few remarks about the subspace \( [K\mu] \), which we define to be the smallest \( B \)-invariant subspace of \( (K,1 \cdot 1) \) containing \( K\mu \).

By the Lebesgue decomposition theorem, the space of measures can be decomposed as \( M = M_\# \oplus M_\# \), where \( M_\# = \{ \varphi dm : \varphi \in L^1(m) \} \) (the absolutely continuous measures with respect to Lebesgue measure \( m \)) and \( M_\#, \{ \kappa \mu : \mu = \kappa \mu \perp m \} \) (the singular ones). Moreover, if \( \mu = \mu_\# + \mu_\# \) (\( \mu_\# = \mu \in M_\# \), \( \mu_\# = \mu_\# \)), then

\[
1 = 1 \mu_\# 1 = 1 \mu_\# + 1 \mu_\#.
\]

As a consequence of this, the space \( (K,1 \cdot 1) \) can be decomposed as \( K = K_\# \oplus K_\# \) where \( K_\# = \{ \kappa dm : \varphi \in L^1(m) \} \) and \( K_\# = \{ \kappa \mu : \mu \perp m \} \). One can show that \( K \perp \nu M/H^\# \) (where \( H^\# \) is equated with a subspace of \( M_\# \) in the obvious way) and \( K_\# \perp \nu L^1 / H^\# \). This makes the space \( (K_\#,1 \cdot 1) \) separable. See [4], [5], and [6] for details.

Although the \( B \)-invariant subspaces of \( (K,1 \cdot 1) \) are very much unknown (due to the nonseparability of \( K_\# \)), the \( B \)-invariant subspaces of \( (K_\#,1 \cdot 1) \) are known [1] (see also [6, p. 99]).

**Theorem 4.2** (Aleksandrov). If \( M \) is a \( B \)-invariant subspace of \( (K_\#,1 \cdot 1) \), then there is an inner function \( \partial \) such that \( f \in M \) if and only if \( f/\partial \) has a pseudocountinuity across \( T \) to a function belonging to \( N^+ (D_\circ) \) and vanishing at infinity.

We now examine \( [K\mu] \) (the smallest \( B \)-invariant subspace of \( (K,1 \cdot 1) \) containing \( K\mu \)), where \( \mu \in M \) and whose support is not all of \( T \). First notice the following.

**Proposition 4.3.** If \( \mu \in M \not\in \{0\} \) with \( \mu \ll m \) and \( \text{supp}(\mu) = T \), then \( [K\mu] = K_\# \).
Proof. Indeed, if the support of $\mu$ omits the arc $J \subseteq T$, then $K\mu$ has an analytic continuation across $J$ given by

$$
\mu(z) = \frac{r}{1 - \zeta z}, \quad z \in D_\theta.
$$

Moreover, if $[K\mu] = K_\alpha$, then by Aleksandrov’s theorem, $K\mu/\theta$ will have a pseudocontinuation for some inner function $\theta$. But since any inner function $\theta$ has a pseudocontinuation given by

$$
\theta(z) = \frac{1}{1 - \bar{\theta}(1/z)}
$$

then $K\mu$ will have a pseudocontinuation $F$. That is to say, $F$ is meromorphic on $D_\theta$ and has nontangential boundary values equal to those of $K\mu$ $m$-almost everywhere on $T$. So there are two meromorphic functions on $D_\theta$, namely $F$ and $\theta$, that have nontangential boundary values equal to $K\mu$ $m$-almost everywhere on the arc $J$. By Privalov’s uniqueness theorem [12, pp. 62 - 63], $F = \theta$. Thus $\mu$ is a pseudocontinuation of $K\mu$ across $T$. So

$$
\lim_{\zeta \rightarrow 1^-} \left( [(K\mu)(\zeta)] - \mu(\zeta) \right) = 0
$$

for $m$-almost every $\zeta$. By Fatou’s jump theorem and the absolute continuity of $\mu$, $\mu$ must be the zero measure, a contradiction.

If $p$ is an analytic polynomial, then $p(B)K\mu = K(p(B)d\mu)$. Assuming $\text{supp}(\mu) = T$, we can apply Mergelyan’s theorem [17, p. 423] along with the density of the continuous functions in $L^1(\mu)$ as well as the inequality $1/K\mu \leq 1/\mu_1$, to conclude that

$$
[K\mu] = \text{cl}_{SC} \{ K(fd\mu) : f \in L^1(\mu) \}. \quad (4.4)
$$

Recall from the definition of the norm and (4.1) that for $\mu \in M_2$, $1/K\mu = 1/\mu_1$. It follows now from (4.4) that for $\mu \bot m$ and $\text{supp}(\mu) = T$,

$$
[K\mu] = \{ K(fd\mu) : f \in L^1(\mu) \}. \quad (4.5)
$$

If $\mu_1 \ll \mu_2$ with $\text{supp}(\mu_2) = T$, then $d\mu_1 = g d\mu_2$, where $g \in L^1(\mu_2)$. Thus if $f \in L^1(\mu_2)$, then $K(fd\mu_1) = K(fd\mu_2)$ and so by (4.4), we have shown the following.

**Proposition 4.6.** If $\mu_1 \ll \mu_2$ and $\text{supp}(\mu_2) = T$, then $[K\mu_1] \subseteq [K\mu_2]$.

If $\mu \in M$ and is positive with $\text{supp}(\mu) = T$, and $\mu = \mu_a + \mu_s$ ($\mu_a \in M_a$ and $\mu_s \in M_s$), we note that $\mu_s \ll \mu$ and $\mu_s \ll \mu$. We can now apply Proposition 4.6 along with (4.5) and Proposition 4.3 to obtain the following result.

**Theorem 4.7.** If $\mu \in M \setminus \{ 0 \}$ is positive with $\text{supp}(\mu) = T$ and $\mu = \mu_a + \mu_s$, then

$$
[K\mu] = K_a \oplus \{ K(fd\mu_s) : f \in L(\mu_s) \} \quad \text{if } \mu_a \equiv 1
$$

$$
\{ K(fd\mu_s) : f \in L(\mu_s) \} \quad \text{if } \mu_a \equiv 0.
$$

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4Privalov’s uniqueness theorem: If $f$ is meromorphic on $D$ and has nontangential limits that exist and are equal to zero on a set of positive measure in $T$, then $f$ is the zero function.
References


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