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THE BACKWARD SHIFT ON THE SPACE OF CAUCHY TRANSFORMS

JOSEPH A. CIMA, ALEC MATHESON, AND WILLIAM T. ROSS

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Abstract. This note examines the subspaces of the space of Cauchy transforms of measures on the unit circle that are invariant under the backward shift operator \( f \to z^{-1}(f - f(0)) \). We examine this question when the space of Cauchy transforms is endowed with both the norm and weak* topologies.

1. Introduction and preliminaries

In this note, we will examine the invariant subspaces of the backward shift operator

\[
(Bf)(z) = \frac{f(z) - f(0)}{z}
\]

on the space of Cauchy transforms \( K \) consisting of analytic functions on the open unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) that take the form

\[
(K\mu)(z) = \frac{d\mu(\zeta)}{1 - \zeta z}
\]

(1.1)

Here \( \mu \in M \), the space of finite Borel measures on the unit circle \( T = \{ z \in \mathbb{C} : |z| = 1 \} \).

By an “invariant subspace” of \( K \) we will mean a closed linear manifold \( M \subseteq K \) for which \( hM \subseteq M \). In using the word “closed”, there are two topologies on \( K \) to consider here. The first is the norm topology. For \( f \in K \), let

\[
M_f := \{ \nu \in M : f = K\nu \}
\]

be the set of “representing measures” for \( f \). Define the norm of an element \( f \in K \) by

\[
1f := \inf \{ 1v1 : v \in M_f \},
\]

where \( 1v1 \) denotes the total variation norm of the measure \( v \). The notation \( (K, 1 \cdot 1) \) will denote the space \( K \) endowed with the above norm topology. It is well known that \( (K, 1 \cdot 1) \) is isometrically isomorphic to the quotient space \( M/H_0^1 \), and is a non-separable Banach space. Here \( H^1 \) is the usual Hardy space of the disk \([9]\) and \( H_0^1 \) are the functions in \( H^1 \) that vanish at the origin. \( H_0^1 \) is regarded as a subspace of \( M \) in the natural way as \( \{ f dm : f \in H^1 \} \) where \( dm = |d\zeta|/2\pi \) is normalized Lebesgue measure on the circle. The second topology on \( K \) is the weak* topology.
that arises by identifying the dual space of the disk algebra $A$ (analytic functions on $D$ that have continuous extensions to $D^*$) with $K$ via the pairing

$$ f \in A, \mu \in M. $$

By the F. and M. Riesz theorem [9, p. 41], if $\mu_1, \mu_2 \in M_K$, then $d\mu_1 - d\mu_2 = \hat{h} dm$, where $h \in H_1$. Thus the above pairing is independent of the representing measure $\mu$. We will use the notation $(K, *)$ to denote the space $K$ endowed with the weak* topology. One can show that $(K, *)$ is separable. Furthermore, every weak* closed subspace of $K$ is norm closed. See [4], [5], and [6] for a review of these basic facts about $K$. In this paper, we examine the $B$-invariant subspaces of $(K, *)$ and $(K, 1 \cdot 1)$.

To put our results in perspective, we mention some known results about the $B$-invariant subspaces for other spaces of analytic functions. For example, by Beurling’s theorem [9, p. 114], the $B$-invariant subspaces of the classical Hardy space $H^2$ all take the form $(\mathcal{D}_g)^\perp$, where $\mathcal{D}_g$ is an inner function. Moreover [8] (see also [6]), $f$ belongs to $(\mathcal{D}_g)^\perp$ if and only if there is a function $G_t \in N_0(D_\theta)$ that vanishes at infinity such that

$$ \lim_{r \to 1^-} f(r\zeta) = \lim_{r \to 1^-} G_t(\zeta/r) $$

for $m$-almost every $\zeta \in T$. Here $D_\theta := \mathcal{D}_g \cap D^-$ and $G_t \in N_0(D_\theta)$ means $G_t(1/\zeta) \in N_+^*$ (the Smirnov class of $D$ [9, p. 25]). The function $G_t$ is called a “pseudocontinuation” of $f$. If

$$ \sigma(\mathcal{D}_g) := \{ \zeta \in D^- : \lim_{\lambda \to \zeta} |\mathcal{D}_g(\lambda)| = 0 \}, $$

then, by basic properties of inner functions [11, pp. 68 and 69], $\mathcal{D}_g$ has an analytic continuation to $\mathcal{D}_g(\mathcal{D}_g)^*$, where $\mathcal{D}_g(\mathcal{D}_g)^* := \{ \zeta \in \mathcal{D}_g : 1/\zeta < 1(\mathcal{D}_g) \}$. In fact, every $f \in (\mathcal{D}_g)^\perp$ has an analytic continuation to $\mathcal{D}_g(\mathcal{D}_g)^*$ [8].

For the Bergman space $L_a^p$ (analytic functions $f$ on $D$ such that $f \in L^p(dx dy)$) a theorem of Richter and Sundberg [14] says that every $B$-invariant subspace takes the form $M_g := \{ f : f \perp \mathcal{D}_g : n \in N \cup \{0\} \}$ for some $g$ in the Dirichlet space (i.e., $g \in L_a^2$). Here we equate the dual of $L_a^p$ with the Dirichlet space via the “Cauchy” dual pairing

$$ \lim_{r \to 1^-} f(r\zeta)g(r\zeta) dm(\zeta). $$

Furthermore, (i) $gM_g \subseteq H^p$ for all $0 < p < 1$, (ii) for every $f \in M_g$, the meromorphic function $f/\partial_g$ (where $\partial_g$ is the inner factor of $g$) has a pseudocontinuation as in (1.2), (iii) every $f \in M_g$ has an analytic continuation to $\mathcal{D}_g(\mathcal{D}_g)^*$. Moreover [2], if $g$ is “sufficiently smooth”, then $gM_g \subseteq H^1$ and $f \in L^2$ belongs to $M_g$ if and only if (a) $g \in H^1$ and (b) $f/\partial_g$ has pseudocontinuation as in (1.2). For certain $L^p$ Bergman spaces, the function $g$ can always be chosen to be “sufficiently smooth”; so in this case we have a complete characterization of the $B$-invariant subspaces. Our purpose here is to get similar-looking results for the space $(K, *)$ (which can be gleaned from results of Korenblum [13]) and to examine the more difficult problem of characterizing the $B$-invariant subspaces of $(K, 1 \cdot 1)$.

$^1$If $h$ is meromorphic on $D$ and $H$ is meromorphic on $D_\theta$ and the nontangential boundary values of $h$ and $H$ exist and are equal $m$-almost everywhere, then $h$ and $H$ are “pseudocontinuations” of each other. See [15] for more details.
2. The main results

For a $B$-invariant subspace $M$ of $(K, *)$ let

$$M_{\perp} = \{ f \in A : (f, K\mu) = 0 \text{ for all } K\mu \in M \}$$

be the pre-annihilator of $M$. $M_{\perp}$ is a norm closed subspace of the disk algebra $A$.

A straightforward calculation shows that

$$\lim_{r \to 1^{-}} \int_{\mathbb{T}} f(\zeta) (K\mu)(r\zeta) dm(\zeta) = \lim_{n \to \infty} \langle \mu, \vartheta^{n} \rangle^{\perp}$$

and $(f, BK\mu) = (f, K(d\mu)) = (\vartheta f, K\mu)$. Thus $zM_{\perp} \subseteq M_{\perp}$ since $BM \subseteq M$.

Since $A$ is a Banach algebra and polynomials are dense in $A$ [11, p. 17], $M_{\perp}$ is an ideal of $A$. A theorem of Rudin [16] (see also [11, p. 85]) says the following.

**Theorem 2.2** (Rudin). Let $I$ be a norm closed ideal of the disk algebra $A$. Then there is a closed set $E \subseteq \mathbb{T}$ of Lebesgue measure zero and an inner function $\vartheta$ with $\sigma(\vartheta) \cap T \subseteq E$ such that

$$I = I(\vartheta, E) := \{ f \in A : f/\vartheta \in H^\infty, f|_{E} = 0 \}.$$ 

Furthermore, given a set $E \subseteq \mathbb{T}$ of Lebesgue measure zero and an inner function $\vartheta$ with $\sigma(\vartheta) \cap \mathbb{T} \subseteq E$, there is an outer function $F \in A$ whose zero set is equal to $E$ and such that $g := \vartheta F$ generates $I(\vartheta, E)$ in the sense that the smallest norm closed ideal of $A$ containing $g$ is equal to $I(\vartheta, E)$.

To describe $M_{\perp}$, we need (via the Hahn-Banach theorem) to describe the set

$$(M_{\perp})^\perp = I(\vartheta, E)^\perp := \{ f \in K : (h, f) = 0 \text{ for all } h \in I(\vartheta, E) \},$$

or equivalently, the set $\{ f \in K : (\vartheta^{n}, f) = 0 \forall n \in \mathbb{N} \cup \{0\} \}$. Korenblum [13] proved the following.

**Theorem 2.3** (Korenblum). If $K\mu \perp I(\vartheta, E)$, then $K\mu$ has an analytic continuation to the set $\mathcal{N}(\sigma(\vartheta)^* \cup E)$.

In the process of proving our main theorem (Theorem 2.5), we will give an alternate proof of Korenblum’s theorem. Any measure $\mu \in M$ can be decomposed uniquely as

$$(2.4) \quad d\mu = \varphi dm + d\mu_{s},$$

where $\varphi \in L^{1}(m)$ and $\mu_{s} \perp m$. Our main theorem describes $I(\vartheta, E)^\perp$.

**Theorem 2.5.** For $\mu \in M$, $K\mu \perp I(\vartheta, E)$ if and only if

1. the support of $\mu_s$ is contained in $E$;
2. $K\mu/\vartheta$ has an analytic continuation across $\mathbb{T} \setminus E$ to a function $F \in N^{1}(D_{\vartheta})$ with $F(\infty) = 0$.

By the F. and M. Riesz theorem, every measure $\nu \in M_{\perp}$ ($f \in K$) has the same singular part. Thus in condition (1), there is only one singular part to consider.

In $H^{2}$, the $B$-invariant subspace $(\vartheta H^{2})^\perp$ is singly generated by the vector $f = B\vartheta$. This next corollary is the analogue of this for $(K, *)$.

**Corollary 2.6.** $I(\vartheta, E)^\perp = \{ B^{n}\varphi : n \in \mathbb{N} \cup \{0\} \}$, where $f = B(K\mu)$ for $d\mu = \vartheta dm + d\mu_{s}$ and $\mu_{s} \perp m$ with support equal to $E$. 


Here \( V \) is the closed linear span in \((K, +)\). This next corollary mimics what happens in the Bergman space setting. By a classical result of Smirnov [9, p. 39], \( K \subseteq \mathcal{H}^p \) for all \( 0 < p < 1 \), and so if our \( B \)-invariant subspace \( M \subseteq K \) has the property that \( \mathcal{M}_\perp \) is generated by \( f \), i.e., \( \mathcal{M}_\perp \) is the closed linear span (in \( A \)) of \( z^nf \ (n \in \mathbb{N} \cup \{ 0 \}) \), then certainly \( fM \subseteq \mathcal{H}^p \) for all \( 0 < p < 1 \). If \( f \) is sufficiently smooth, we get the stronger condition \( fM \subseteq H^1 \) and even a bit more.

**Theorem 2.7.** Suppose \( f \in A \) with \( f' \in H^\infty \). Let \( E = f^{-1}(\{ 0 \}) \cap T \), and let \( \mathcal{F} \) be the inner factor of \( f \). Then \( K \mu \perp z^nf \) for all \( n \in \mathbb{N} \cup \{ 0 \} \) if and only if

1. \( fK \mu \in H^1 \);
2. \( K \mu \mathcal{F} \) has an analytic continuation across \( T \setminus E \) to a function \( F \in N^\perp(D_0) \) with \( F(\infty) = 0 \).

If \( f \in A \) with \( f' \in H^\infty \), then the boundary zero set \( E \) of \( f \) satisfies the so-called Carleson condition: If \( (l_n) \) is the sequence of arcs contiguous to \( E \) on the circle, then \( f' \mid l_n \mid \log |l_n| > -\infty \). Thus, by Theorem 2.2, not every \( B \)-invariant subspace of \((K, +)\) is singly generated by such an \( f \).

Comments about the \( B \)-invariant subspaces of \((K, 1 \cdot 1)\) appear at the end of this note.

### 3. The proofs

**Proposition 3.1.** Suppose \( \mathcal{F} \) is a generator for \( I(\mathcal{F}, E) \) and \( du = \varphi dm + d\mu_s \) as in \((2.4)\). Then \( K \mu \perp z^\mathcal{F} F \) for all \( n \in \mathbb{N} \cup \{ 0 \} \) if and only if \( \varphi \in \mathcal{H}_0^\perp \) and \( \mu_s \) is supported in \( E \).

**Proof.** Suppose \( K \mu \perp z^\mathcal{F} F \) for all \( n \in \mathbb{N} \cup \{ 0 \} \). Then, by \((2.1)\),

\[
(3.2) \quad z^\mathcal{F} F (\varphi dm + d\mu_s) = 0 \quad \text{for all } n \in \mathbb{N} \cup \{ 0 \}.
\]

From the F. and M. Riesz theorem, \( \mathcal{F} \) \( d\mu_s \) is the zero measure (and so \( \mu_s \) is supported in \( E \)) and \( \mathcal{F} \varphi = \lambda \in \mathcal{H}_0^\perp \). However, \( \varphi \mathcal{F} = \mathcal{H} \varphi \in \mathcal{H}_0^\perp \) and has \( L^1(m) \) boundary values, and so \( \varphi \mathcal{F} \in \mathcal{H}_0^\perp \) [9, p. 28]. The converse is obvious.

**Proof of Theorem 2.5.** We start by proving a somewhat weaker result: \( K \mu \perp I(E, \mathcal{F}) \) if and only if \( \mu_s \) is supported in \( E \) and \( K \mu \mathcal{F} \) has a pseudocountinuation across \( T \) belonging to \( N^\perp(D_0) \) and vanishing at infinity. Indeed, suppose \( K \mu \perp I(E, \mathcal{F}) \). By Proposition 3.1 we can assume \( \mu \) takes the form

\[
(3.3) \quad d\mu = \varphi dm + d\mu_s. \quad \varphi \mathcal{F} = \lambda \in \mathcal{H}_0^\perp \supp(\mu_s) \subseteq E.
\]

Since \( \lambda \in \mathcal{H}_0^\perp \), then \( \lambda(1/z) \) belongs to \( H^1(D_0) \) and vanishes at infinity. The inner function \( \mathcal{F} \) is defined on \( D_0 \) by \( \mathcal{F}(z) = 1/\lambda(1/z) \). The function

\[
(3.3) \quad T \mu, \mathcal{F}(z) := \mathcal{F}(1/z) + \frac{\mu(z)}{\varphi(z)}, \quad z \in D_0
\]

belongs to \( \mathcal{H}^p(D_0) \) for all \( 0 < p < 1 \) [9, p. 39] and so the function
belongs to $N^*(D_0)$ and vanishes at infinity. By Fatou's jump theorem\(^3\), the boundary function for $T_{\mu,\theta}$ is

$$\frac{\varphi}{\mathcal{G}}(\zeta) + \frac{(K\mu)(\zeta) - \varphi(\zeta)}{\mathcal{G}(\zeta)} = \frac{K\mu}{\mathcal{G}}(\zeta)$$

for $m$-almost every $\zeta \in T$. Thus $T_{\mu,\theta}$ is the pseudocontinuation of $K\mu/\mathcal{G}$ of the desired type.

Conversely, suppose $d\mu = \varphi d\mathcal{H} + d\mu_\varphi$, where $\varphi \in L^1(m)$ and $\mu_\varphi$ is supported in $E$, and $K\mu/\mathcal{G}$ has a pseudocontinuation $G \in N^*(D_0)$ with $G(\infty) = 0$. Then, by Fatou's jump theorem,

$$G(\zeta) = \lim_{r \to 1^-} \frac{K\mu}{\mathcal{G}}(r\zeta) = \frac{\varphi(\zeta) + \varphi(\zeta)}{\mathcal{G}(\zeta)}.$$

Assuming for the moment that $\mathcal{G}(0) = 0$, we conclude that $G - \varphi/\mathcal{G} \in N^*(D_0)$ and vanishes at infinity. Then $\varphi/\mathcal{G}$ is the boundary function of a function from $N^*(D_0)$ that vanishes at infinity. But since $\varphi/\mathcal{G} \in L^1(m)$, then $\varphi/\mathcal{G} \in H^1$. If $\mathcal{G}(0) = 0$, then use the same argument with $\mathcal{G}$ replaced by $\mathcal{G}/\mathcal{G}$ and $G$ replaced by $G/\mathcal{G}$ for some positive integer $n$. Now apply Proposition 3.1.

Now we need to show that $K\mu$ has an analytic continuation to $\mathcal{H}_I^*(\mathcal{G}) \cup E$.

As mentioned earlier, this was originally shown by Korenblum in [13]. Indeed, if $W \subseteq \mathcal{H}_I^*(\mathcal{G}) \cup E$ is an open set containing an arc of the circle, then $T_{\mu,\theta}$ (as defined in (3.3)) is analytic on $W \cap D_0$ and by standard estimates,

$$|T_{\mu,\theta}(\lambda)| \leq C1\mu 1 \frac{1}{|\lambda| - 1}, \lambda \in W \cap D_0.$$

Since $K\mu \perp I(\mathcal{G}, E)$, we can apply Proposition 3.1 to conclude that $\mu$ takes the form

$$d\mu = \varphi d\mathcal{H} + d\mu_\varphi,$$

where $\varphi = \mathcal{H}_I(h \in H^1)$ and $\mu_\varphi$ is supported in $E$.

Next, let $(h_n)$ be a sequence of polynomials in $H^1$ that approximates $h$ in norm and set

$$d\mu_n := \mathcal{H}_I d\mathcal{H} + d\mu_\varphi.$$

Notice that $I\mu_n I$ is uniformly bounded in $n$. By Proposition 3.1, $K\mu_n \perp I(\mathcal{G}, E)$ and the corresponding pseudocontinuation of $K\mu_n/\mathcal{G}$ is

$$T_{\mu_n,\theta}(z) = \frac{1}{\mathcal{G}(z)} + \frac{1}{\mathcal{G}(z)} \frac{d\mathcal{H}_I h(\zeta)}{d\mathcal{H}_I h(\zeta)} dm(\zeta) + \frac{1}{\mathcal{G}(z)} d\mathcal{H}_I d\mathcal{H}_I d\mathcal{H}_I \frac{d\mu_\varphi(\zeta)}{d\mathcal{H}_I h(\zeta)}.$$

Since the functions $\mathcal{H}_I h_n$ are bounded on $T$, then $K\mu_n/\mathcal{G}$ and $T_{\mu_n,\theta}$ are $H^1$ functions on $W \cap D$ and $W \cap D_0$ (respectively) [9, p. 41]. (Note that $\mathcal{G}$ has an analytic continuation across $W \cap T$ as does $\mu_\varphi$ since this $W \cap T$ avoids the support of $\mu_\varphi$.) Moreover, by what was said earlier, they have equal boundary values almost everywhere on $W \cap T$. By a standard Morera's theorem argument [10, p. 95], these two functions are analytic continuations of each other across $W \cap T$.

\(^3\)Fatou's jump theorem: $\lim_{r \to 1^-} |(\mathcal{G}(r\zeta) - \mathcal{G}(\zeta))| = \lim_{r \to 1^-} |\mathcal{G}(d\mu/dm(\zeta))| m$-almost everywhere [9, p. 4].
Finally,
\[
\begin{align*}
&\left| \frac{K_{\mu}}{g}(\lambda) \right| \leq C \frac{1}{1-|\lambda|}, \\
&\left| T_{\mu,\theta}(\lambda) \right| \leq C \frac{1}{1-|\lambda|}.
\end{align*}
\]
\[\lambda \in W \cap D, \quad \lambda \in W \cap D_\theta.\]

By a theorem of Beurling [7] (see also [15, p. 95]), these functions form a normal family on \(W\) and so a subsequence (for which we will use the same index) converges to an analytic function on \(W\). But since \(K_{\mu}\) converges pointwise to \(K_{\mu/\theta}\), then \(K_{\mu/\theta}\), and hence \(K_{\mu}\), will have an analytic continuation to \(W\).

**Proof of Corollary 2.6.** If \(BF\) is a generator of the ideal \(I(\theta,E)\), then by our “Cauchy pairing” in (2.1), it is routine to show that
\[
(z^m BF, B^n f) = r^{-1} \int_{z} \bar{\theta} (\vartheta m + d\mu_\theta) = 0 \quad \forall m, n \in \mathbb{N} \cup \{0\}.
\]
Thus
\[
\{B^n f : n = 0, 1, 2, \ldots \} \subseteq I(\theta,E)^\perp.
\]

If \(g \in A\) satisfies \((g, B^n f) = 0\) for all \(n\), one can use the F. and M. Riesz theorem to show that \(g/\theta \in H^1\) and \(g\) is zero on the support of \(\mu_\theta\) (which equals \(E\)). Thus \(g \in I(\theta,E)\). An application of the Hahn-Banach theorem completes the proof.

The proof of Theorem 2.7 requires a few preliminaries. Notice that \(K_{\mu} \in L^1(dA)\), where \(dA\) is the area measure on \(D\). This follows from Fubini’s theorem and the fact that the integral
\[
\int_{D} \frac{1}{|e^{\theta} - z|} dA(z)
\]
is uniformly bounded in \(\theta\).

For a Cauchy transform \(K_{\mu}\), consider the function
\[
(K_{\mu})(z), \quad \lambda \in D.
\]
Since \(K_{\mu} \in L^1(dA)\) and is analytic on \(D\), it is not difficult to show, using the fact that \((z - \lambda)^{-1} \in L^1(dA)\) for each fixed \(\lambda \in D\), that the above integral exists for every \(\lambda \in D\). Moreover, the dominated convergence theorem says that the above function is continuous on \(D\).

**Proposition 3.4.** For \(\mu \in M\),
\[
\sup_{0 < r < 1} \int_{D} \frac{1}{z - re^{\theta}} dA(z) \, d\theta < \infty.
\]

**Proof.** For fixed \(0 < r < 1\),
\[
\int_{D} \frac{1}{z - re^{\theta}} dA(z) \, d\theta
\]
\[
\int_{D} \frac{1}{|e^{i\theta} - a|} d\theta.\]

Use the standard inequality
\[
\int_{0}^{2\pi} \frac{d\theta}{|e^{\theta} - a|} \leq C \log(\frac{1}{1 - |a|}), \quad |a| < 1
\]
to get
\[ r \int_0^{2\pi} \frac{d\theta}{|se^{it} - re^{i\theta}|} \begin{cases} \frac{1}{r} \log(\frac{1}{1-s/r}) & \text{for } s < r, \\ \frac{1}{r} \log(\frac{1}{1-r/s}) & \text{for } s > r. \end{cases} \]

and
\[ r \int_0^{2\pi} \frac{dt}{|1 - se^{it}e^{-ix}|} C \log(\frac{1}{1-s}). \]

Combine the above two estimates along with Fubini's theorem to show the desired integral is bounded above by
\[ C \int_0^r \log(\frac{1}{1-s}) \log(\frac{1}{1-s/r}) ds + \int_r^1 \log(\frac{1}{1-s}) \log(\frac{1}{1-r/s}) ds. \]

Standard estimates now show that this quantity is bounded uniformly for \( r \) close to 1.

**Proof of Theorem 2.7.** Suppose \( f \in A \) with \( f^l \in H^\infty \) and \( K_\mu \bot z^nf \) for all \( n \in \mathbb{N} \cup \{0\} \). Theorem 2.5 yields condition (2). Using a power series argument, one can show that
\[ (f, K_\mu) = \lim_{r \to 1^-} \int_D \frac{(K_\mu)(r)(zf)(z)}{r} \, dm_2(z), \]
where \( dm_2 = dA/\pi \). Since \((zf)^l\) is a bounded function, we can use the fact that \( K_\mu \in L^1(dA) \), to rewrite this as
\[ \int_D (K_\mu)(z)(zf)(z) \, dm_2(z). \]

For fixed \( \lambda \in D \), the function
\[ K_\mu - (K_\mu)(\lambda) \]
\[ \frac{1}{z - \lambda} \]
can be written as \( K_{\lambda \mu} \), where \( d\mu = \frac{1}{\lambda - \lambda} d\mu \). By Proposition 3.1, \( K_{\lambda \mu} \) also annihilates the ideal generated by \( f = \Re F \). Thus, by what was said above,
\[ (K_\mu)(z) - (K_\mu)(\lambda) \]
\[ \frac{1}{z - \lambda} \int_D (zf)(z) \, dm_2(z) = 0, \quad \lambda \in D. \]

Another power series computation yields
\[ \int_D \frac{(zf)(z)}{z - \lambda} \, dm_2(z) = -\frac{(zf)(\lambda)}{\lambda} \]
and so from (3.5),
\[ -\frac{(zf)(\lambda)}{\lambda}(K_\mu)(\lambda) = \int_D \frac{(K_\mu)(z)}{z - \lambda} (zf)(z) \, dm_2(z). \]

Now use Proposition 3.4 along with the assumption that \((zf)^l\) is bounded to show that the integrals
\[ r \int_0^{2\pi} [f(re^{i\theta})(K_\mu)(re^{i\theta})] \, d\theta \]
are uniformly bounded in \( 0 < r < 1 \), that is to say, \( fK_\mu \in H^1 \).

---

\(^3\)See, for example, the argument used to prove Lemma 2.5 in [3].
Conversely, suppose conditions (1) and (2) are satisfied. Since \( f_0 K \) and \( F_t \) (where \( F_t \) is the outer factor of \( f \)) are the boundary values of functions from \( N^\ast(D_\sigma) \), then \( f_0 K \) is also the boundary function of a \( N^\ast(D_\sigma) \) function that vanishes at infinity. But since \( f_0 K \in L^1(m) \), then \( f_0 K \in H_0^1 \). Thus

\[
(K_\mu)(\zeta)\beta(\zeta) dm(\zeta) = 0 \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}.
\]

Finally, using the notation \( g_\zeta(z) := g(rz) \),

\[
(K_\mu)\beta r - K_\mu r = (K_\mu)\beta r - K_\mu f = \quad \frac{f_r}{f} - \frac{f_r}{f} + K_\mu f = \quad \frac{f_r}{f} - \frac{f_r}{f} = \frac{f_r}{f} - \frac{f_r}{f}.
\]

which goes to zero in the \( L^1(m) \) norm as \( r \to 1^- \). Thus for any \( n \in \mathbb{N} \cup \{0\} \),

\[
(\zeta^n f, K_\mu) = \lim_{r \to 1^-} (K_\mu)(\zeta)\beta(\zeta)\beta(\zeta) dm(\zeta) = (K_\mu)(\zeta)\beta(\zeta) dm(\zeta) = 0.
\]

4. The norm topology

Recall that \((K,1 \cdot 1)\) is a nonseparable space, and so a characterization of the \( B \)-invariant subspaces is out of reach. In this final section, we will make a few remarks about the subspace \([K_\mu]\), which we define to be the smallest \( B \)-invariant subspace of \((K,1 \cdot 1)\) containing \( K_\mu \).

By the Lebesgue decomposition theorem, the space of measures can be decomposed as \( M = M_a \oplus M_s \), where \( M_a = \{ \phi dm : \phi \in L^1(m) \} \) (the absolutely continuous measures with respect to Lebesgue measure \( m \)) and \( M_s = \{ \mu \in M : \mu \perp m \} \) (the singular ones). Moreover, if \( \mu = \mu_a + \mu_s \) (\( \mu_a \in M_a, \mu_s \in M_s \)), then

\[
(1 \mu 1) = 1 \mu_a 1 + 1 \mu_s 1.
\]

As a consequence of this, the space \((K,1 \cdot 1)\) can be decomposed as \( K = K_a \oplus K_s \) where \( K_a = \{ K(\phi dm) : \phi \in L^1(m) \} \) and \( K_s = \{ K_\mu : \mu \perp m \} \). One can show that \( K \) is equated with a subspace of \( M_a \) in the obvious way and \( K_s \) is isomorphic to \( L^1/H_0^1 \). This makes the space \((K_a,1 \cdot 1)\) separable. See [4], [5], and [6] for details.

Although the \( B \)-invariant subspaces of \((K,1 \cdot 1)\) are very much unknown (due to the nonseparability of \( K_s \)), the \( B \)-invariant subspaces of \((K_a,1 \cdot 1)\) are known [1] (see also [6, p. 99]).

**Theorem 4.2** (Aleksandrov). If \( M \) is a \( B \)-invariant subspace of \((K_a,1 \cdot 1)\), then there is an inner function \( \theta \) such that \( f \in M \) if and only if \( f/\theta \) has a pseudocontinuation across \( T \) to a function belonging to \( N^\ast(D_\sigma) \) and vanishing at infinity.

We now examine \([K_\mu]\) (the smallest \( B \)-invariant subspace of \((K,1 \cdot 1)\) containing \( K_\mu \)), where \( \mu \in M \) and whose support is not all of \( T \). First notice the following.

**Proposition 4.3.** If \( \mu \in M \setminus \{0\} \) with \( \mu \ll m \) and \( \supp(\mu) \cap T \neq \emptyset \), then \([K_\mu] = K_a\).
Proof. Indeed, if the support of \( \mu \) omits the arc \( J \subseteq T \), then \( K\mu \) has an analytic continuation across \( J \) given by
\[
\mu(z) = \frac{d\mu(\zeta)}{1 - \zeta z}, \quad z \in D_e.
\]
Moreover, if \( [K\mu] \neq K_\vartheta \), then by Aleksandrov's theorem, \( K\mu/\vartheta \) will have a pseudocontinuation for some inner function \( \vartheta \). But since any inner function \( \vartheta \) has a pseudocontinuation given by
\[
\vartheta(z) = \frac{1}{\zeta \in D_e}
\]
then \( K\mu \) will have a pseudocontinuation \( F \). That is to say, \( F \) is meromorphic on \( D_e \) and has nontangential boundary values equal to those of \( K\mu \) almost everywhere on \( T \). So there are two meromorphic functions on \( D_e \), namely \( F \) and \( \vartheta \) that have nontangential boundary values equal to \( K\mu \) almost everywhere on the arc \( J \). By Privalov's uniqueness theorem [12, pp. 62 - 63], \( F = \vartheta \). Thus \( \mu \) is a pseudocontinuation of \( K\mu \) across \( T \). So
\[
\lim_{r \to 1^-} \frac{[(K\mu)(r\zeta) - \mu(\zeta/r)]}{r} = 0
\]
for \( \mu \) almost every \( \zeta \). By Fatou's jump theorem and the absolute continuity of \( \mu \), \( \mu \) must be the zero measure, a contradiction.

If \( p \) is an analytic polynomial, then \( p(B)K\mu = K(p(\zeta d\mu)) \). Assuming \( \text{supp}(\mu) = T \), we can apply Mergelyan's theorem [17, p. 423] along with the density of the continuous functions in \( L^1(\mu) \) as well as the inequality \( IK\mu1\mu1 \), to conclude that
\[
[K\mu] = \text{clos}_K \{K(fd\mu) : f \in L^1(\mu) \}.
\]
Recall from the definition of the norm and (4.1) that for \( \mu \in M_\vartheta, 1K\mu1 = 1\mu1 \). It follows now from (4.4) that for \( \mu \perp m \) and \( \text{supp}(\mu) = T \),
\[
[K\mu] = \{K(fd\mu) : f \in L^1(\mu) \}.
\]
If \( \mu_1 \ll \mu_2 \) with \( \text{supp}(\mu_2) = T \), then \( d\mu_1 = g d\mu_2 \), where \( g \in L^1(\mu_2) \). Thus if \( f \in L^1(\mu_1) \), then \( K(fd\mu_1) = K(gd\mu_2) \) and so by (4.4), we have shown the following.

Proposition 4.6. If \( \mu_1 \ll \mu_2 \) and \( \text{supp}(\mu_2) = T \), then \( [K\mu_1] \subseteq [K\mu_2] \).

If \( \mu \in M \) and is positive with \( \text{supp}(\mu) = T \), and \( \mu = \mu_a + \mu_s \) (\( \mu_a \in M_a \) and \( \mu_s \in M_s \)), we note that \( \mu_a \ll \mu \) and \( \mu_s \ll \mu \). We can now apply Proposition 4.6 along with (4.5) and Proposition 4.3 to obtain the following result.

Theorem 4.7. If \( \mu \in M \setminus \{0\} \) is positive with \( \text{supp}(\mu) = T \) and \( \mu = \mu_a + \mu_s \), then
\[
[K\mu] = \begin{cases} 
K_a \oplus \{K(fd\mu_s) : f \in L^1(\mu_s) \} & \text{if } \mu_a \equiv 1 \\
0 & \text{if } \mu_a \equiv 0 
\end{cases}
\]

Privalov's uniqueness theorem: If \( f \) is meromorphic on \( D \) and has nontangential limits that exist and are equal to zero on a set of positive measure in \( T \), then \( f \) is the zero function.
References

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