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ZEROS OF FUNCTIONS WITH FINITE DIRICHLET INTEGRAL

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Abstract. In this paper, we refine a result of Nagel, Rudin, and Shapiro (1982) concerning the zeros of holomorphic functions on the unit disk with finite Dirichlet integral.

This is a remark about the zeros of functions \( f = \sum_{n=0}^{\infty} a_n z^n \) holomorphic on \( U \) that have finite Dirichlet integral

\[
D(f) := \frac{1}{\pi} \int_U |f|^2 \, dA = \sum_{n=0}^{\infty} |a_n|^2,
\]

where \( dA \) is Lebesgue measure in the plane. Clearly such functions belong to the classical Hardy space \( H^2 \), and so the zeros \( \{z_n\}_n \subset U \) of \( f \) (repeated according to multiplicity) satisfy the Blaschke condition \( n \log(1 - |z_n|) \to \infty \) [4, p. 18]. However, not every Blaschke sequence are the zeros of a holomorphic \( f \) with \( D(f) < \infty \) [2].

In 1962, Shapiro and Shields [6] improved a result of Carleson [3] and showed that if

\[
1 \log(1 - |z_n|) < \infty
\]

then there is a nontrivial holomorphic \( f \) on \( U \) with \( D(f) < \infty \) such that \( f(z_n) = 0 \) for all \( n \).

This condition does not completely characterize the zero sets of analytic functions with finite Dirichlet integral. For example, if \( \{z_n\}_n \subset (0, 1) \) is a Blaschke sequence for which (1) fails, then \( f = (1 - z)^2 \) has finite Dirichlet integral, where \( B \) is the Blaschke product with zeros \( \{z_n\}_n \). Nevertheless, in the converse direction, Nagel, Rudin, and Shapiro [5] proved that if \( \{r_n\}_n \subset (0, 1) \) is such that

\[
\sum_{n=0}^{\infty} -\log(1 - r_n) = \infty,
\]

then there is a sequence of angles \( \{\theta_n\}_n \) such that the sequence \( \{r_n e^{i\theta_n}\}_n \) is not the zeros of any nontrivial holomorphic function \( f \) on \( U \) with \( D(f) < \infty \). They do this by first noting that when \( D(f) \to \infty \), the limit

\[
\lim_{z \to e^{i\theta} \in \Omega_{r_n} \neq \emptyset} f(z)
\]

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exists for almost every $e^{i\theta}$, where $\Omega_{e^{i\theta}}$ is the exponential contact region

$$\Omega_{e^{i\theta}} := \{ re^{i\theta} : 1 - r^2 > e^{-\pi/n} \}.$$ 

Beginning at $z = 1$, lay down arcs $I_n \subset \partial U$ of length

$$\frac{1}{-\log(1 - r_n)}$$

e to end (repeatedly traversing the unit circle). Since $\int_{\partial U} |I_n| = \infty$, by hypothesis, each $e^{i\theta} \in \partial U$ will be contained in infinitely many of the intervals $(I_n)_{n \in \mathbb{N}}$. Let $e^{i\theta}$ be the center of the interval $I_n$, and note that simple geometry shows that for every $e^{i\theta}$, the exponential contact region $\Omega_{e^{i\theta}}$ contains infinitely many of the points $r e^{i\theta}$. Thus if $f$ has finite Dirichlet integral and $f(r e^{i\theta}) = 0$ for all $n$, the above limit result says that the boundary function for $f$ will vanish almost everywhere on $\partial U$, forcing $f$ to be identically zero. This argument actually shows that the sequence $(r e^{i\theta})_{n \in \mathbb{N}}$ cannot be the zeros of a nontrivial harmonic function $f$ on $U$ with finite Dirichlet integral (where $|f|^2$ is replaced by $|\nabla f|^2$ in the definition of the Dirichlet integral).

In this note, we refine this result and show that for analytic functions the angles $\theta_n$ can be chosen so that the zeros $(r e^{i\theta_n})_{n \in \mathbb{N}}$ need not accumulate at every point of the circle, but instead accumulate at a single point.

**Theorem 2.** Suppose $(r_n)_{n \in \mathbb{N}} \subset (0, 1)$ with $r_n \to 1$ and

$$\lim_{n \to \infty} \frac{1}{-\log(1 - r_n)} = \infty.$$ 

Then there are angles $(\theta_n)_{n \in \mathbb{N}}$ such that $\text{clos}(r_n e^{i\theta_n})_{n \in \mathbb{N}} \cap \partial U = \{1\}$ and such that if $f$ is holomorphic on $U$ with $D(f) < \infty$ and $f(r_n e^{i\theta_n}) = 0$ for all $n$, then $f$ is identically the zero function.

Our proof is based on the following lemma. In order to make our construction easier, we will work in the upper half plane.

**Lemma 3.** Let $J \subset \mathbb{R}$ be a finite open interval with center $x_0$ and $0 < y_0 < |J|$. Set

$$S := \{ x + iy : x \in J, \ 0 < y < |J| \}$$

and $\lambda_0 := x_0 + iy_0$. Suppose $f$ is holomorphic on $S$ with

$$\int_S |f|^2 \, dA < \infty$$

and $f(\lambda_0) = 0$. If $E = \{ x \in J : |f(x)| > 1 \}$, then

$$\int_S |f|^2 \, dA \leq \frac{c}{\log(|J|/y_0)},$$

where $c$ is a universal constant.

**Proof.** Elementary considerations show that $\omega_{\lambda_0}^S(E)$, the harmonic measure of $E$ with respect to $S$ at $\lambda_0$, is bounded below by a universal constant times $y_0/|J|$. Indeed,

$$\omega_{\lambda_0}^S(E) \geq \omega_{\lambda_0}^S(F).$$
where $F$ is the union of two intervals in the real line of length $\frac{1}{2} |J|$ located at the lower corners of $S$. Let $\psi : S \to U$ be the conformal map that takes the centroid of $S$ to the origin and the line segment containing $z_0$ and $x_0$ to the positive real axis. Thus $\psi(x_0) = 1$ and $\psi(z_0) = r$ with $1 - r \cap \gamma_0 / |J|$. Then

$$\omega^S_{h_0}(F) = \left| \psi(F) \right| \cap \omega_r(\psi(F)) = \frac{1 - r^2}{r |e^{\psi(F)} - r|^2} dt.$$ 

But since $\psi(F)$ is a fixed distance from the point $z = 1$, the denominator in the above integral does not matter. Thus, since the measure of $\psi(F)$ is fixed, the above integral is comparable to $1 - r \cap \gamma_0 / |J|$. 

Let $\phi : U \to S$ be the conformal map with $\phi(0) = z_0$, and let $g := f \circ \phi$. Then $g(0) = 0$, $|g| \leq 1$ on $\phi^{-1}(E)$, and $|\phi^{-1}(E)| = \omega_{h_0}^S(E) \cap \gamma_0 / |J|$. This means that $g$ is a “test function” for the logarithmic capacity of $\phi^{-1}(E)$ [1, Theorem 2], and so

$$\int_U |g|^2 dA \cap \psi \cap (\phi^{-1}(E)) \cap \log(|J| / \gamma_0).$$

Here we are using the well-known fact that if $W \subset \partial U$ with $a = |W|$, then

$$\text{cap}(W) \cap \frac{c}{\log(1/a)}.$$ 

Finally, note that

$$\int_U |g|^2 dA = \int_S |f|^2 dA.$$

We are now ready to prove our main theorem. To make the construction easier, we work in the upper half plane and replace the sequence $(r_n)$ with a sequence $(y_n) \subset (0, 1)$ with $y_n \to 0$ and such that

$$\lim_{n \to \infty} \frac{1}{\log(1/y_n)} = \infty.$$ 

We will construct a sequence $(x_n + iy_n)$ in the upper half plane whose closure intersects the real axis only at $x = 0$ and such that the only holomorphic function $f$ in the upper half plane with finite Dirichlet integral for which $f(x_n + iy_n) = 0$ for all $n$ is the zero function.

Assuming that $y_n \to 0$, we can find

$$1 \cap n_1 \cap m_1 < n_2 < m_2 < \cdots$$

such that, whenever $n \cap m_k$,

$$y_n \log \frac{1}{y_n} < \frac{1}{k^2} \cap e^{-2k^2}$$

and

$$ke^{2k^2} < \frac{1}{\log(1/y_n)} < ke^{2k^2} + 1.$$ 

For each $k$, lay out intervals

$$J_{n_k}, J_{n_k+1}, \cdots, J_{m_k}$$

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on the real axis end-to-end starting at \( x = 0 \) and such that

\[
|J_n| = \frac{1}{k^2 e^{2k^2} \log(1/y_n)} n_k n mk.
\]

Then

\[
\log \frac{|J_n|}{y_n} = \log \frac{1}{k^2 e^{2k^2} y_n \log(1/y_n)} = \log(1/y_n) - \log k^2 - 2k^2 - \log \log(1/y_n) < \log(1/y_n).
\]

Let \( x_n \) be the center of \( J_n \) and set \( \lambda_n := x_n + iy_n \) and

\[
S_n := \{ x + iy : x \in J_n, 0 < y < |J_n| \}.
\]

Suppose that \( f \) is holomorphic on the upper half plane with finite Dirichlet integral and such that \( f(\lambda_n) = 0 \) for all \( n \) \( n \) \( m_k \). Set

\[
A_k := \{ n : n \) \( m_k \text{ and } |f| e^{-k^2} \text{ on a set } E_n \subset J_n \text{ with } |E_n| \) \( k |J_n| \}.
\]

\[
B_k := \{ n : n \) \( m_k, n / \in A_k \}.
\]

Apply Lemma 3 to see that if \( n \in A_k \), then

\[
\int_{S_n} |f|^2 dA \) \( c e^{-k^2} \text{ } 1 \text{ } e^{-2k^2} \frac{1}{\log(|J_n|/y_n)} \text{ } \frac{1}{\log(1/y_n)} c\).
\]

and if \( n \in B_k \), then

\[
\int_{J_n} \frac{1}{|f|} dx \) \( \frac{1}{2} |J_n| \) \( k = \frac{1}{2} e^{-2k^2} \frac{1}{\log(1/y_n)} c\).
\]

We conclude that

\[
\sum_{n \in A_k} |f|^2 dA + \sum_{n \in B_k} \int_{J_n} \frac{1}{|f|} dx \) \( \frac{1}{2} e^{-2k^2} \frac{1}{\log(1/y_n)} c\).
\]

Thus by the log-integrability of \( f \) on the boundary [4, p. 17], \( f \) must be the zero function.

It follows that the set

\[
(\lambda_n) n \) \( m_k \)
\]

cannot be the zeros of a holomorphic function with finite Dirichlet integral. Choose the remaining points (from the unused \( y_n \)'s) on the imaginary axis to obtain a sequence \( (\lambda_n) n \) \( m_k \) that is not the zero set of a function with finite Dirichlet integral. Finally, since

\[
|J_n| = \frac{1}{k^2 e^{2k^2} \log(1/y_n)} n_k n mk \leq \frac{k^2 e^{2k^2} + 1}{k^2 e^{2k^2}} \to 0, \text{ } k \to \infty,
\]

it follows that the closure of the sequence \( (\lambda_n) n \) \( m_k \) intersects the real axis only at \( x = 0 \).
References


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