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ZEROS OF FUNCTIONS WITH FINITE DIRICHLET INTEGRAL

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(Communicated by Juha M. Heinonen)

Abstract. In this paper, we refine a result of Nagel, Rudin, and Shapiro (1982) concerning the zeros of holomorphic functions on the unit disk with finite Dirichlet integral.

This is a remark about the zeros of functions \( f = \sum_{n=0}^{\infty} a_n z^n \) holomorphic on \( U = \mathbb{D} \) that have finite Dirichlet integral

\[
D(f) := \frac{1}{\pi} \int_U |f|^2 dA = \sum_{n=0}^{\infty} |a_n|^2,
\]

where \( dA \) is Lebesgue measure in the plane. Clearly such functions belong to the classical Hardy space \( H^2 \), and so the zeros \((z_n)_{n=0}^{\infty} \subset U \) of \( f \) (repeated according to multiplicity) satisfy the Blaschke condition \( r_n (1 - |z_n|^2) < \infty \). However, not every Blaschke sequence are the zeros of a holomorphic \( f \) with \( D(f) < \infty \).

In 1962, Shapiro and Shields [6] improved a result of Carleson [3] and showed that if

\[
\frac{1}{\sum_{n=1}^{\infty} -\log(1 - |z_n|)} < \infty,
\]

then there is a nontrivial holomorphic \( f \) on \( U \) with \( D(f) < \infty \) such that \( f(z_n) = 0 \) for all \( n \).

This condition does not completely characterize the zero sets of analytic functions with finite Dirichlet integral. For example, if \((z_n)_{n=0}^{\infty} \subset (0, 1)\) is a Blaschke sequence for which (1) fails, then \( f = (1 - z)^2 \) has finite Dirichlet integral, where \( B \) is the Blaschke product with zeros \((z_n)_{n=0}^{\infty}\). Nevertheless, in the converse direction, Nagel, Rudin, and Shapiro [5] proved that if \((r_n)_{n=0}^{\infty} \subset (0, 1)\) is such that

\[
\frac{1}{\sum_{n=0}^{\infty} -\log(1 - r_n)} = \infty,
\]

then there is a sequence of angles \((\theta_n)_{n=0}^{\infty}\) such that the sequence \((r_n e^{i\theta_n})_{n=0}^{\infty}\) is not the zeros of any nontrivial holomorphic function \( f \) on \( U \) with \( D(f) < \infty \). They do this by first noting that when \( D(f) < \infty \), the limit

\[
\lim_{z \to e^{i\theta}, z \in \Omega_{\delta \theta}} f(z)
\]
exists for almost every \( e^{i\theta} \), where \( \Omega_{e^{i\theta}} \) is the exponential contact region

\[
\Omega_{e^{i\theta}} := \{ r e^{i\theta} : 1 - r^2 > e^{-\pi/2 - \pi r} \},
\]

Beginning at \( z = 1 \), lay down arcs \( I_n \subset \partial U \) of length

\[
\frac{1}{-\log(1 - r_n)}
\]
end-to-end (repeatedly traversing the unit circle). Since \( \int_0^1 |I_n| = \infty \), by hypothesis, each \( e^{i\theta} \in \partial U \) will be contained in infinitely many of the intervals \( (I_n)_{n=0}^{\infty} \). Let \( e^{i\theta} \) be the center of the interval \( I_n \), and note that simple geometry shows that for every \( e^{i\theta} \), the exponential contact region \( \Omega_{e^{i\theta}} \) contains infinitely many of the points \( r_n e^{i\theta} \). Thus if \( f \) has finite Dirichlet integral and \( f(r_n e^{i\theta}) = 0 \) for all \( n \), the above limit result says that the boundary function for \( f \) will vanish almost everywhere on \( \partial U \), forcing \( f \) to be identically zero. This argument actually shows that the sequence \( (r_n e^{i\theta})_{n=0}^{\infty} \) cannot be the zeros of a nontrivial harmonic function \( f \) on \( U \) with finite Dirichlet integral (where \(|f|^2 \) is replaced by \(|\nabla f|^2 \) in the definition of the Dirichlet integral).

In this note, we refine this result and show that for analytic functions the angles \( \theta_n \) can be chosen so that the zeros \( (r_n e^{i\theta_n})_{n=0}^{\infty} \) need not accumulate at every point of the circle, but instead accumulate at a single point.

**Theorem 2.** Suppose \( (r_n)_{n=0}^{\infty} \subset (0, 1) \) with \( r_n \to 1 \) and

\[
\lim_{n\to\infty} \frac{1}{-\log(1 - r_n)} = \infty.
\]

Then there are angles \( (\theta_n)_{n=0}^{\infty} \) such that \( \text{clos}(r_n e^{i\theta_n})_{n=0}^{\infty} \cap \partial U = \{1\} \) and such that if \( f \) is holomorphic on \( U \) with \( D(f) < \infty \) and \( f(r_n e^{i\theta_n}) = 0 \) for all \( n \), then \( f \) is identically the zero function.

Our proof is based on the following lemma. In order to make our construction easier, we will work in the upper half plane.

**Lemma 3.** Let \( J \subset \mathbb{R} \) be a finite open interval with center \( x_0 \) and \( 0 < y_0 < |J| \). Set

\[
S := \{ x + iy : x \in J, \ 0 < y < |J| \}
\]
and \( \lambda_0 := x_0 + iy_0 \). Suppose \( f \) is holomorphic on \( S \) with

\[
\int_S |f|^2 dA < \infty
\]
and \( f(\lambda_0) = 0 \). If

\[
E = \{ x \in J : |f(x)| > 1 \},
\]
and \( |E| < \frac{1}{2} |J| \), then

\[
\int_S |f|^2 dA \geq \frac{c}{\log(|J/y_0|)},
\]
where \( c \) is a universal constant.

**Proof.** Elementary considerations show that \( \omega^S_{\lambda_0}(E) \), the harmonic measure of \( E \) with respect to \( S \) at \( \lambda_0 \), is bounded below by a universal constant times \( y_0/|J| \).

Indeed,

\[
\omega^S_{\lambda_0}(E) \geq \omega^S_{\lambda_0}(F).
\]
where $F$ is the union of two intervals in the real line of length $\frac{1}{2} J$ located at the lower corners of $S$. Let $\psi : S \to U$ be the conformal map that takes the centroid of $S$ to the origin and the line segment containing $\lambda_0$ and $x_0$ to the positive real axis. Thus $\psi(x_0) = 1$ and $\psi(\lambda_0) = r$ with $1 - r \neq y_0/|J|$. Then
\[
\omega^\mathcal{D}_{\lambda_0}(F) = |\psi(F)| \omega^\mathcal{D}_r(\psi(F)) = \frac{1 - r^2}{|\psi(F)| |e^r - r|^2} dt.
\]
But since $\psi(F)$ is a fixed distance from the point $z = 1$, the denominator in the above integral does not matter. Thus, since the measure of $\psi(F)$ is fixed, the above integral is comparable to $1 - r \neq y_0/|J|$.

Let $\phi : U \to S$ be the conformal map with $\phi(0) = \lambda_0$, and let $g := f \circ \phi$. Then $g(0) = 0$, $|g| \geq 1$ on $\phi^{-1}(E)$, and $|\phi^{-1}(E)| = \omega^\mathcal{D}_{\lambda_0}(E) \psi y_0/|J|$. This means that $g$ is a “test function” for the logarithmic capacity of $\phi^{-1}(E)$ [1, Theorem 2], and so
\[
\int_U |g|^2 dA \leq c \text{ cap}(\phi^{-1}(E)) \leq \frac{c}{\log(|J|/y_0)}.
\]
Here we are using the well-known fact that if $W \subset \partial U$ with $a = |W|$, then
\[
\text{cap}(W) \leq \frac{c}{\log(1/a)}.
\]
Finally, note that
\[
\int_U |g|^2 dA = \int_S |f|^2 dA.
\]

We are now ready to prove our main theorem. To make the construction easier, we work in the upper half plane and replace the sequence $(r_n)_{n \in \mathbb{N}}$ with a sequence $(y_n)_{n \in \mathbb{N}}$ in $(0, 1)$ with $y_n \to 0$ and such that
\[
\lim_{n \to \infty} \frac{1}{\log(1/y_n)} = \infty.
\]
We will construct a sequence $(x_n + iy_n)_{n \in \mathbb{N}}$ in the upper half plane whose closure intersects the real axis only at $x = 0$ and such that the only holomorphic function $f$ in the upper half plane with finite Dirichlet integral for which $f(x_n + iy_n) = 0$ for all $n$ is the zero function.

Assuming that $y_n \to 0$, we can find
\[
1 \neq n_1 < m_1 < n_2 < m_2 < \cdots
\]
such that, whenever $n \neq n_k$,
\[
y_n \log \frac{1}{y_n} < \frac{1}{k^2} e^{-2k^2}
\]
and
\[
ke^{2k^2} < \frac{1}{\log(1/y_n)} < ke^{2k^2} + 1.
\]
For each $k$, lay out intervals
\[J_{n_k}, J_{n_k+1}, \ldots, J_m\]
on the real axis end-to-end starting at \( x = 0 \) and such that
\[
|J_n| = \frac{1}{k^2 e^{2k^2} \log(1/n)}, \quad nk \in n \otimes m_k.
\]

Then
\[
\log \frac{|J_n|}{yn} = \log \frac{1}{k^2 e^{2k^2} yn \log(1/yn)} = \log(1/yn) - \log k^2 - 2k^2 - \log(1/yn) < \log(1/yn).
\]

Let \( x_n \) be the center of \( J_n \) and set \( \lambda_n := x_n + iy_n \) and
\[
S_n := \{ x + iy : x \in J_n, 0 < y < |J_n| \}.
\]

Suppose that \( f \) is holomorphic on the upper half plane with finite Dirichlet integral and such that \( f(\lambda_n) = 0 \) for all \( nk \in n \otimes m_k \). Set
\[
A_k := \{ n : nk \in n \otimes m_k \} \text{ and } |f| \in L^2(\mathbb{R}^+; J_n) \text{ on a set } E_n \subset J_n \text{ with } |E_n| \in L^1(J_n).
\]

By Lemma 3, if \( n \in A_k \), then
\[
\log \frac{1}{|f|} dx = \frac{1}{2} |J_n| k^2 = \frac{1}{2} \frac{e^{-2k^2}}{\log(1/yn)}
\]

and if \( n \in B_k \), then
\[
\log \frac{1}{|f|} dx = \frac{1}{2} |J_n| k^2 = \frac{1}{2} \frac{e^{-2k^2}}{\log(1/yn)}
\]

We conclude that
\[
\sum_{n \in A_k} |f|^2 dA + \sum_{n \in B_k} \int_{J_n} \log \frac{1}{|f|} dx = \sum_{n=0}^{m_k} \frac{e^{-2k^2}}{\log(1/yn)} \otimes k.
\]

Thus by the log-integrability of \( f \) on the boundary [4, p. 17], \( f \) must be the zero function.

It follows that the set
\[
(\lambda_n)_{n \in n \otimes m_k}
\]

cannot be the zeros of a holomorphic function with finite Dirichlet integral. Choose the remaining points (from the unused \( y_n \)'s) on the imaginary axis to obtain a sequence \( (\lambda_n)_{n \otimes m_k} \) that is not the zero set of a function with finite Dirichlet integral. Finally, since
\[
\frac{m_k}{n=0} \left| J_n \right| = \frac{1}{k^2 e^{2k^2}} \frac{1}{\log(1/yn)} \leq \frac{ke^{2k^2} + 1}{k^2 e^{2k^2}} 
\]

it follows that the closure of the sequence \( (\lambda_n)_{n \otimes m_k} \) intersects the real axis only at \( x = 0 \).
References


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