2008

Truncated Toeplitz Operators on Finite Dimensional Spaces

William T. Ross
University of Richmond, wross@richmond.edu

Joseph A. Cima

Warren R. Wogen

Recommended Citation

Cima, Joseph A., William T. Ross, and Warren R. Wogen. "Truncated Toeplitz Operators on Finite Dimensional Spaces." *Operators and Matrices* 2, no. 3 (2008): 357-369. doi:10.7153/oam-02-21.

TRUNCATED TOEPLITZ OPERATORS ON FINITE DIMENSIONAL SPACES

JOSEPH A. CIMA, WILLIAM T. ROSS, AND WARREN R. WOGEN

Abstract. In this paper, we study the matrix representations of compressions of Toeplitz operators to the finite dimensional model spaces $H^2 \ominus BH^2$, where B is a finite Blaschke product. In particular, we determine necessary and sufficient conditions - in terms of the matrix representation - of when a linear transformation on $H^2 \ominus BH^2$ is the compression of a Toeplitz operator. This result complements a related result of Sarason [6].

1. Introduction

If H^2 is the classical *Hardy space* of the open unit disk $\mathbb{D} := \{|z| < 1\}$ and P is the orthogonal projection of $L^2 = L^2(\partial\mathbb{D}, d\theta/2\pi)$ onto H^2 (see [3] for the basic definitions), one defines for $\varphi \in L^\infty$ the *Toeplitz operator* T_φ on H^2 by the formula

$$T_\varphi f = P(\varphi f).$$

Recently, Sarason [6] initiated a study of *truncated Toeplitz operators.* These are operators A_φ defined on the model spaces

$$K_\vartheta := H^2 \cap (\vartheta H^2)^\perp,$$

where ϑ is an inner function, by the formula

$$A_\varphi f := P_\vartheta(\varphi f).$$

Here P_ϑ is the orthogonal projection of L^2 onto K_ϑ. In other words, A_φ is the compression of T_φ to K_ϑ. In [6, Thm. 4.1], the set

$$\mathcal{T}_\vartheta := \{A_\varphi : \varphi \in L^2 \text{ and } A_\varphi \text{ is bounded}\}$$

is characterized as follows: A bounded operator A on K_ϑ belongs to $\mathcal{T}_\vartheta$ if and only if there are functions $g_1, g_2 \in K_\vartheta$ such that

$$A = A_z^* A A_z + g_1 \otimes k + k \otimes g_2, \tag{1.1}$$

where

$$k(z) := \frac{\vartheta(z) - \vartheta(0)}{z}$$

and $h_1 \otimes h_2$ denotes the rank-one operator

$$h_1 \otimes h_2(f) = \langle f, h_2 \rangle h_1.$$

Though the condition in (1.1) determines which bounded operators on K_ϑ belong to $\mathcal{T}_\vartheta$, it is difficult to apply since it depends on the *existence* of the functions $g_1, g_2 \in K_\vartheta$. In this paper we will obtain, in the special case when K_ϑ is finite

2000 *Mathematics Subject Classification.* Primary 47B35; Secondary 47B32, 30D55.
Key words and phrases. Toeplitz operators, model spaces, Clark operators, matrix representations.

dimensional, a more tangible condition. A finite n-dimensional model space takes the form K_B, where B is a finite Blaschke product with zeros $\{a_1, \cdots, a_n\}$. It is well known that K_B consists of all functions of the form

$$f(z) = \frac{p(z)}{\prod_{j=1}^{n}(1 - \overline{a_j} z)} \tag{1.2}$$

where p is any polynomial of degree at most $n-1$. Furthermore,

$$k_\lambda(z) := \frac{1 - \overline{B(\lambda)}B(z)}{1 - \overline{\lambda} z} \tag{1.3}$$

is the reproducing kernel for K_B in that $k_\lambda \in K_B$ for all $\lambda \in \mathbb{D}$ and

$$f(\lambda) = \langle f, k_\lambda \rangle \quad \forall f \in K_B.$$

In the above formula, the inner product is the L^2 inner product

$$\langle f, g \rangle = \int_{\mathbb{T}} f(\zeta)\overline{g(\zeta)} \frac{|d\zeta|}{2\pi},$$

where $\mathbb{T} := \partial\mathbb{D}$. Using (1.2) and interpolating, it is easy to show that given distinct points $\lambda_1, \cdots, \lambda_n \in \mathbb{D}$, the set $\{k_{\lambda_1}, \cdots, k_{\lambda_n}\}$ is a basis for K_B. If we assume that the zeros $a_1, \cdots, a_n$ of B are distinct, a (non-orthonormal) basis for K_B is $\{k_{a_1}, \cdots, k_{a_n}\}$ where

$$k_{a_j}(z) = \frac{1}{1 - \overline{a_j} z}.$$

By elementary linear algebra, the complex vector space of all linear transformations on K_B has dimension n^2. By Sarason [6, Thm. 7.1], $\mathcal{T}_B$ has dimension $2n-1$. This leads to the natural question as to which linear transformations on K_B belong to $\mathcal{T}_B$. Our first theorem is the following.

Theorem 1.4. *Let B be a finite Blaschke product of degree n with distinct zeros $a_1, \cdots, a_n$ and let A be any linear transformation on the n-dimensional space K_B. If $M_A = (r_{i,j})$ is the matrix representation of A with respect to the basis $\{k_{a_1}, \cdots, k_{a_n}\}$, then $A \in \mathcal{T}_B$ if and only if*

$$r_{i,j} = \overline{\left(\frac{B'(a_1)}{B'(a_i)}\right)} \left(\frac{r_{1,i}\overline{(a_1 - a_i)} + r_{1,j}\overline{(a_j - a_1)}}{\overline{a_j - a_i}}\right), \quad 1 \leqslant i, j \leqslant n, i \neq j. \tag{1.5}$$

Remark 1.6.
1. Theorem 1.4 says that the matrix representation of a truncated Toeplitz operator is determined by the entries along the main diagonal and the first row. Notice how such matrices have dimension $2n-1$ as they should since $\mathcal{T}_B$ has dimension $2n-1$.
2. There is nothing special about the first row. For example, a similar result can be obtained where the representing matrix is determined by the entries along the main diagonal and the first column.
3. The proof of this theorem will also yield an algorithm for determining the symbol φ from the matrix entries.
4. When $n = 2$, the matrix

$$\begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{pmatrix}$$

is the matrix representation of a truncated Toeplitz operator with respect to the basis $\{k_{a_1}, k_{a_2}\}$ if and only if

$$\overline{B'(a_1)} r_{1,2} = r_{2,1} \overline{B'(a_2)}.$$

Although $\{k_{a_1}, \cdots, k_{a_n}\}$ is a natural basis for K_B, it is not an orthonormal one. An important orthonormal basis for K_B is the *Clark basis* $\{v_{\zeta_1}, \cdots, v_{\zeta_n}\}$ which are the normalized eigenvectors corresponding to the eigenvalues $\zeta_j \in \mathbb{T}$ for the *Clark unitary operator* U_α where $\alpha \in \mathbb{T}$. This is formed as follows: Since B is a finite Blaschke product, it is analytic in an open neighborhood of $\mathbb{D}^-$ and hence, for each $\zeta \in \mathbb{T}$, the kernel function k_ζ defines an analytic function on $\mathbb{D}$. It is routine to show that $k_\zeta \in K_B$ and

$$f(\zeta) = \langle f, k_\zeta \rangle \quad \forall f \in K_B. \tag{1.7}$$

For each $\alpha \in \mathbb{T}$ a routine exercise, using the fact that B' never vanishes on $\mathbb{T}$, will show that there are exactly n distinct points $\zeta_1, \cdots, \zeta_n \in \mathbb{T}$ for which

$$B(\zeta_j) = \frac{\alpha + B(0)}{1 + \overline{B(0)}\alpha}, \quad j = 1, \cdots, n.$$

Another routine exercise will show that

$$\|k_{\zeta_s}\|^2 = |B'(\zeta_s)|$$

and so we form the normalized kernel functions

$$v_{\zeta_s} := \frac{k_{\zeta_s}}{\sqrt{|B'(\zeta_s)|}}. \tag{1.8}$$

The points $\zeta_1, \cdots, \zeta_n$ turn out to be the eigenvalues of the Clark unitary operator,

$$U_\alpha := A_z + \frac{B(0) + \alpha}{1 - |B(0)|^2} (k_0 \otimes \widetilde{k}_0) \tag{1.9}$$

with corresponding eigenvectors $v_{\zeta_1}, \cdots, v_{\zeta_n}$. (Here and for what follows below,

$$\widetilde{k}_\lambda(z) := \frac{B(z) - B(\lambda)}{z - \lambda}. \tag{1.10}$$

One can show [6] that $\widetilde{k}_\lambda \in K_B$ for all $\lambda \in \mathbb{D}$.) Thus $\{v_{\zeta_1}, \cdots, v_{\zeta_n}\}$ is an orthonormal basis for K_B. The operators U_α, first explored by Clark [2], have been well studied and generalized [1, 5]. By the spectral theorem, we know that the matrix representation of U_α with respect to this basis is $\operatorname{diag}(\zeta_1, \cdots, \zeta_n)$. Our next theorem replaces the basis $\{k_{a_1}, \cdots, k_{a_n}\}$ of kernel functions with the Clark basis $\{v_{\zeta_1}, \cdots, v_{\zeta_n}\}$.

Theorem 1.11. *Suppose B is a finite Blaschke product of degree n and $\alpha \in \mathbb{T}$. Let A be any linear transformation on the n-dimensional space K_B. If $M_A = (r_{i,j})$ is the matrix representation of A with respect to the Clark basis $\{v_{\zeta_1}, \cdots, v_{\zeta_n}\}$ corresponding to α, then $A \in \mathcal{T}_B$ if and only if*

$$r_{i,j} = \frac{\sqrt{|B'(\zeta_1)|}}{\zeta_j - \zeta_i} \left(\frac{\zeta_j}{\zeta_i} \frac{1}{\sqrt{|B'(\zeta_j)|}} (\zeta_1 - \zeta_i) r_{1,i} + \frac{1}{\sqrt{|B'(\zeta_i)|}} (\zeta_j - \zeta_1) r_{1,j} \right) \tag{1.12}$$

for all $1 \leqslant i, j \leqslant n, i \neq j$.

Remark 1.13. (1) Exactly as in the previous theorem, the matrix representation of a truncated Toeplitz operator is determined by the entries along the main diagonal and the first row.

(2) The proof of this theorem will also yield an algorithm for determining the symbol φ from the matrix entries.
(3) When $n = 2$, the matrix

$$\begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{pmatrix}$$

is the matrix representation of a truncated Toeplitz operator with respect to the basis $\{v_{\zeta_1}, v_{\zeta_2}\}$ if and only if

$$\zeta_1 r_{1,2} = \zeta_2 r_{2,1}.$$

If we alter the basis $\{v_{\zeta_1}, \cdots, v_{\zeta_n}\}$ slightly, we get even more. Indeed, let

$$\beta_\alpha := \frac{\alpha + B(0)}{1 + \overline{B(0)}\alpha}, \quad w_s := e^{-\frac{i}{2}(\arg(\zeta_s) - \arg(\beta_\alpha))}, \quad e_{\zeta_s} := \frac{1}{\sqrt{|B'(\zeta_s)|}} w_s k_{\zeta_s}. \tag{1.14}$$

Garcia and Putinar in [4] show that $\{e_{\zeta_1}, \cdots, e_{\zeta_n}\}$ is an orthonormal basis which not only diagonalizes the Clark operator U_α but has the additional property that the matrix representation of *any* truncated Toeplitz operator with respect to this basis is complex symmetric. A matrix M is *complex symmetric* if $M = M^t$, where t denotes the transpose. This next theorem replaces the Clark basis $\{v_{\zeta_1}, \cdots, v_{\zeta_n}\}$ with this new basis $\{e_{\zeta_1}, \cdots, e_{\zeta_n}\}$.

Theorem 1.15. *Suppose B is a finite Blaschke product of degree n and $\alpha \in \mathbb{T}$. Let A be any linear transformation on the n-dimensional space K_B. If $M_A = (r_{i,j})$ is the matrix representation of A with respect to the basis $\{e_{\zeta_1}, \cdots, e_{\zeta_n}\}$ corresponding to α, then $A \in \mathcal{T}_B$ if and only if M_A is complex symmetric and*

$$r_{i,j} = \frac{\sqrt{|B'(\zeta_1)|}}{\overline{w_1}} \frac{1}{\zeta_j - \zeta_i} \left(\frac{\overline{w_j}}{\sqrt{|B'(\zeta_j)|}} (\zeta_1 - \zeta_i) r_{1,i} + \frac{\overline{w_i}}{\sqrt{|B'(\zeta_i)|}} (\zeta_j - \zeta_1) r_{1,j} \right) \tag{1.16}$$

for all $1 \leqslant i, j \leqslant n, i \neq j$.

Remark 1.17. When $n = 2$, the theorem says that *any* complex symmetric 2×2 matrix represents a truncated Toeplitz operator with respect to the basis $\{e_{\zeta_1}, e_{\zeta_2}\}$. This was previously observed by Sarason [6, §5].

In [6, §12], Sarason began a discussion on how the Clark unitary operators somehow generate the truncated Toeplitz operators (see Remark 3.3 below). In finite dimensions, we have the following result.

Theorem 1.18. *Let B be a Blaschke product of degree n and let $\alpha_1, \alpha_2 \in \mathbb{T}$ with $\alpha_1 \neq \alpha_2$. Then for any $\varphi \in L^2$, there are polynomials p, q of degree at most n so that*

$$A_\varphi = p(U_{\alpha_1}) + q(U_{\alpha_2}). \tag{1.19}$$

Remark 1.20. (1) Sarason in [6, Thm. 10.1] proves that $p(U_\alpha) \in \mathcal{T}_B$ for every polynomial p and every $\alpha \in \mathbb{T}$. In fact, Theorem 1.18 can be gleaned from the proof of Thm. 7.1 in [6] along with the spectral theorem for unitary operators.
(2) We will see in Remark 3.3 that, in a certain sense, one can compute the polynomials p and q in (1.19) from φ.

2. Proof of Theorem 1.4

For a given $\varphi \in L^2$, decompose φ as

$$\varphi = \psi_1 + \overline{\psi_2} + \eta_1 + \overline{\eta_2}, \quad \psi_1, \psi_2 \in K_B, \quad \eta_1, \eta_2 \in BH^2.$$

Now write A_φ as

$$A_\varphi = A_{\psi_1 + \overline{\psi_2}} + A_{\eta_1 + \overline{\eta_2}}$$

and notice from [6, Thm. 3.1] that the second term on the right is zero. Thus

$$\left\{ A_{\psi_1 + \overline{\psi_2}} : \psi_1, \psi_2 \in K_B \right\} = \mathcal{T}_B. \tag{2.1}$$

We are assuming that the zeros $a_1, \cdots, a_n$ of B are distinct and so the functions

$$\widetilde{k}_{a_j}(z) = \frac{B(z)}{z - a_j}, \quad j = 1, \cdots, n$$

form a basis for K_B and

$$\overline{\widetilde{k}_{a_j}(z)} = \overline{\left(\frac{B(z)}{z - a_j}\right)}, \quad j = 1, \cdots, n$$

form a basis for $\overline{K_B}$.

From the above discussion and (2.1), $\mathcal{T}_B$ consists of A_φ, where

$$\varphi(\zeta) = \sum_{j=1}^{n} c_j \overline{\left(\frac{B(\zeta)}{\zeta - a_j}\right)} + \sum_{j=1}^{n} d_j \frac{B(\zeta)}{\zeta - a_j} \tag{2.2}$$

and c_j, d_j are arbitrary complex numbers. Combine this with the identity

$$k_\lambda \otimes \widetilde{k}_\lambda = A_{\overline{\frac{B}{z - \lambda}}}. \tag{2.3}$$

[6, Thm. 7.1] and its adjoint to see that $\mathcal{T}_B$ consists of operators of the form

$$\sum_{j=1}^{n} c_j k_{a_j} \otimes \widetilde{k}_{a_j} + \sum_{j=1}^{n} d_j \widetilde{k}_{a_j} \otimes k_{a_j}, \tag{2.4}$$

where c_j, d_j are complex numbers.

In a moment, we will find the matrix representation of the above operator with respect to the basis $\{k_{a_1}, \cdots, k_{a_n}\}$. Before doing this, we need a few formulas. Using the reproducing property of k_{a_j} and the definitions of k_{a_j} (1.3) and $\widetilde{k}_{a_j}$ (1.10) we obtain

$$\langle \widetilde{k}_{a_i}, k_{a_j} \rangle = \begin{cases} 0, & \text{if } i \neq j; \\ B'(a_j), & \text{if } i = j. \end{cases} \quad \text{and} \quad \langle \widetilde{k}_{a_i}, \widetilde{k}_{a_j} \rangle = \frac{1}{1 - \overline{a_j} a_i}. \tag{2.5}$$

We know, since $\{k_{a_1}, \cdots, k_{a_n}\}$ is a basis for K_B, that

$$\widetilde{k}_{a_j} = \sum_{s=1}^{n} h_s(a_j) k_{a_s}$$

for some complex constants $h_s(a_j)$. Using (2.5) one can compute $h_s(a_j)$ and get

$$\widetilde{k}_{a_j} = \sum_{s=1}^{n} \frac{1}{\overline{B'(a_s)}} \frac{1}{1 - \overline{a_s} a_j} k_{a_s}. \tag{2.6}$$

We are now ready for the proof in Theorem 1.4. Let A_φ be of the form in (2.4) and let

$$(b_{s,p})_{1\leqslant s,p\leqslant n} = M_{A_\varphi}$$

be the matrix representation of A_φ with respect to the basis $\{k_{a_1}, \cdots, k_{a_n}\}$. We want to show that

$$b_{s,p} = \overline{\left(\frac{B'(a_1)}{B'(a_s)}\right)} \left(\frac{b_{1,s}\overline{(a_1 - a_s)} + b_{1,p}\overline{(a_p - a_1)}}{\overline{a_p - a_s}}\right), \quad 1 \leqslant s, p \leqslant n, s \neq p. \tag{2.7}$$

A computation with (2.4), (2.5), and (2.6) will show that

$$A_\varphi k_{a_p} = c_p\overline{B'(a_p)}k_{a_p} + \sum_{s=1}^{n}\left(\frac{1}{\overline{B'(a_s)}}\sum_{j=1}^{n}\frac{d_j}{(1-\overline{a_s}a_j)(1-\overline{a_p}a_j)}\right)k_{a_s}.$$

Thus

$$b_{s,p} = c_p\overline{B'(a_p)}\delta_{s,p} + \frac{1}{\overline{B'(a_s)}}\sum_{j=1}^{n}\frac{d_j}{(1-\overline{a_s}a_j)(1-\overline{a_p}a_j)}.$$

The identities in (2.7) follow from the formula

$$\frac{1}{(1-\overline{a_s}a_j)(1-\overline{a_p}a_j)} = \frac{-\overline{a_s}}{\overline{a_p}-\overline{a_s}}\frac{1}{1-\overline{a_s}a_j} + \frac{\overline{a_p}}{\overline{a_p}-\overline{a_s}}\frac{1}{1-\overline{a_p}a_j}.$$

One direction of the proof is now complete.

For the other direction, let V denote the set of all matrices satisfying (1.5). These identities show that each $M = (r_{i,j}) \in V$ is determined uniquely by its entries along the diagonal and the first row. Furthermore, M is a linear function of these entries. It follows that V is a $2n-1$ dimensional vector space. We have already shown via (2.7) that

$$V_1 := \{M_{A_\varphi} : A_\varphi \in \mathcal{T}_B\} \subset V$$

and, from Sarason's theorem, V_1 has dimension $2n-1$. Therefore, $V_1 = V$. The proof is now complete.

Remark 2.8. (1) Note that $\{D_1, \cdots, D_n, R_2, \cdots, R_n\}$ is an explicit basis for V. Here $D_k = \operatorname{diag}(0, \cdots, 1, 0, \cdots, 0)$ and R_k is the matrix satisfying (1.5) with $r_{1,k} = 1$, $r_{1,j} = 0$ if $j \neq k$, and $r_{j,j} = 0$ for all j. For example, if $n = 3$, then

$$D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R_2 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{B'(a_1)}{B'(a_2)} & 0 & \frac{(a_1-a_2)B'(a_1)}{(a_3-a_2)B'(a_2)} \\ 0 & \frac{(a_2-a_1)B'(a_1)}{(a_2-a_3)B'(a_3)} & 0 \end{pmatrix}^*$$

$$R_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{(a_3-a_1)B'(a_1)}{(a_3-a_2)B'(a_2)} \\ \frac{B'(a_1)}{B'(a_3)} & \frac{(a_1-a_3)B'(a_1)}{(a_2-a_3)B'(a_3)} & 0 \end{pmatrix}^*$$

In the above, $*$ denotes complex conjugation of all the entries of the matrix (not the conjugate transpose).

(2) If
$$P_j := \frac{1}{B'(a_j)} k_{a_j} \otimes \widetilde{k}_{a_j},$$
notice from the above proof that
$$P_j^2 = P_j, \quad \sum_{j=1}^{n} P_j = I, \quad P_j P_l = \delta_{j,l} P_j, \quad \mathcal{T}_B = \operatorname{span}\{P_j, P_j^* : j = 1, \cdots, n\}.$$
Similar identities hold for
$$P_j^* = \frac{1}{B'(a_j)} \widetilde{k}_{a_j} \otimes k_{a_j}.$$
These identities exhibit the linear dependence of the set of $2n$ operators $\{P_j, P_j^* : j = 1, \cdots, n\}$. A little work will show, for example, that the set $\{P_j, P_l^* : j = 2, \cdots, n; l = 1, \cdots, n\}$ forms a basis for $\mathcal{T}_B$ consisting of rank one idempotents.

(3) Using similar techniques, one can compute A_φ from (2.4) with respect to the basis $\{\widetilde{k}_{a_1}, \cdots, \widetilde{k}_{a_n}\}$. In this case, the $b_{s,p}$ entry of this matrix is
$$b_{s,p} = d_p B'(a_p)\delta_{s,p} + \frac{1}{B'(a_s)} \sum_{j=1}^{n} \frac{c_j}{(1 - a_s\overline{a_j})(1 - a_p\overline{a_j})}$$
and the necessary and sufficient condition for a matrix $(r_{s,p})$ to represent (with respect to the basis $\{\widetilde{k}_{a_1}, \cdots, \widetilde{k}_{a_n}\}$) something from $\mathcal{T}_B$ is
$$r_{s,p} = \frac{B'(a_1)}{B'(a_s)} \left(\frac{r_{1,s}(a_1 - a_s) + r_{1,p}(a_p - a_1)}{a_p - a_s} \right), \quad 1 \leqslant s, p \leqslant n, s \neq p.$$

3. Proof of Theorem 1.18

The following lemma can be gleaned from [6, Thm. 7.1]. We include a proof here.

Lemma 3.1. *Suppose $w_1, \cdots, w_{2n-1}$ are distinct points of $\mathbb{T}$. Then the rank-one operators*
$$k_{w_1} \otimes k_{w_1}, \cdots, k_{w_{2n-1}} \otimes k_{w_{2n-1}}$$
are linearly independent.

Proof. Suppose $c_1, \cdots, c_{2n-1}$ are complex constants such that
$$\sum_{j=1}^{2n-1} c_j k_{w_j} \otimes k_{w_j} = 0. \tag{3.2}$$
Since $k_{w_1}, \cdots, k_{w_n}$ are linearly independent, there is a $g \in K_B$ such that
$$\langle k_{w_1}, g \rangle = 1, \quad \langle k_{w_j}, g \rangle = 0, \quad j = 2, \cdots, n.$$
Apply to this g the operator on the left hand side of (3.2) to see that
$$c_1 k_{w_1} + \sum_{j=n+1}^{2n-1} c_j \langle g, k_{w_j} \rangle k_{w_j} = 0.$$
However, the vectors $k_{w_1}, k_{w_{n+1}}, \cdots, k_{w_{2n-1}}$ are linearly independent and so $c_1 = 0$. Now take an appropriate g to show that $c_2 = 0$ and so on. ∎

Here is the proof of Theorem 1.18. Let $\alpha_1, \alpha_2 \in \mathbb{T}$ with $\alpha_1 \neq \alpha_2$. Let $\zeta_1, \cdots, \zeta_n$ and $\eta_1, \cdots, \eta_n$ be the points in $\mathbb{T}$ so that

$$B(\zeta_j) = \beta_{\alpha_1} := \frac{\alpha_1 + B(0)}{1 + \overline{B(0)}\alpha_1}, \quad B(\eta_j) = \beta_{\alpha_2} := \frac{\alpha_2 + B(0)}{1 + \overline{B(0)}\alpha_2}, \quad j = 1, \cdots, n.$$

Notice that the points $\zeta_1, \cdots, \zeta_n, \eta_1, \cdots, \eta_n$ are *distinct* and recall that $\{v_{\zeta_1}, \cdots, v_{\zeta_n}\}$ is a orthonormal basis for K_B of eigenvectors of U_{α_1}. In a similar way, $\{v_{\eta_1}, \cdots, v_{\eta_n}\}$ is a orthonormal basis for K_B of eigenvectors of U_{α_2}. Let

$$P_{\zeta_j} := v_{\zeta_j} \otimes v_{\zeta_j}, \quad P_{\eta_j} := v_{\eta_j} \otimes v_{\eta_j}, \quad j = 1, \cdots, n$$

and observe that these operators are the orthogonal projections onto the eigenspaces spanned by k_{ζ_j} (respectively k_{η_j}). In [6, Thm. 5.1] it was shown, for any $\zeta \in \mathbb{T}$, that

$$k_\zeta \otimes k_\zeta = A_{k_\zeta + \overline{k_\zeta} - 1}$$

and so these projections P_{ζ_j}, P_{η_j} also belong to $\mathcal{T}_B$. Furthermore, by the spectral theorem for unitary operators, we have, for any analytic polynomials p and q,

$$p(U_{\alpha_1}) = \sum_{j=1}^{n} p(\zeta_j) v_{\zeta_j} \otimes v_{\zeta_j}, \quad q(U_{\alpha_2}) = \sum_{j=1}^{n} q(\eta_j) v_{\eta_j} \otimes v_{\eta_j}.$$

and so $p(U_{\alpha_1}), q(U_{\alpha_2}) \in \mathcal{T}_B$.

Then, to show that

$$\mathcal{T}_B = \bigvee \{(U_{\alpha_1})^i, (U_{\alpha_2})^j, 1 \leqslant i, j \leqslant n\},$$

it suffices to prove that

$$\mathcal{T}_B = \bigvee \{P_{\zeta_j}, P_{\eta_j} : j = 1, \cdots, n\},$$

which follows directly from Lemma 3.1 and the fact that $\mathcal{T}_B$ has dimension $2n - 1$.

Remark 3.3. (1) Theorem 1.18 says that any A_φ takes the form $p(U_{\alpha_1}) + q(U_{\alpha_2})$ for some polynomials p and q. We remark here that we can determine p and q from the symbol φ provided it is chosen in a particular way. To see how to do this, notice in the proof of Theorem 1.18, how we have shown that

$$\bigvee \{k_{\zeta_j} \otimes k_{\zeta_j}, k_{\eta_j} \otimes k_{\eta_j} : j = 1, \cdots, n\} = \mathcal{T}_B.$$

In fact any $2n - 1$ of these will form a basis for $\mathcal{T}_B$. But since

$$k_\zeta \otimes k_\zeta = A_{k_\zeta + \overline{k_\zeta} - 1},$$

every operator in $\mathcal{T}_B$ can be written as A_φ where

$$\varphi = \sum_{j=1}^{n} c_j (k_{\zeta_j} + \overline{k_{\zeta_j}} - 1) + \sum_{j=1}^{n} d_j (k_{\eta_j} + \overline{k_{\eta_j}} - 1).$$

Choose polynomials p and q of degree at most n for which

$$p(\zeta_j) = \sqrt{|B'(\zeta_j)|} c_j, \quad q(\eta_j) = \sqrt{|B'(\eta_j)|} d_j, \quad j = 1, \cdots, n.$$

Then we have

$$A_\varphi = p(U_{\alpha_1}) + q(U_{\alpha_2}).$$

Indeed, from the spectral theorem,

$$p(U_{\alpha_1}) = \sum_{j=1}^{n} p(\zeta_j) v_{\zeta_j} \otimes v_{\zeta_j}, \quad q(U_{\alpha_2}) = \sum_{j=1}^{n} q(\eta_j) v_{\eta_j} \otimes v_{\eta_j}.$$

The result now follows.

(2) Sarason [6, § 12] began a discussion on how the Clark unitary operators generate $\mathcal{T}_\vartheta$ for a general inner function ϑ. He used the Clark theory and some recent results of Aleksandrov and Poltoratski to prove, for a bounded Borel function φ and an inner function ϑ, the following integral formula:

$$A_\varphi = \int_{\mathbb{T}} \varphi(U_\alpha) \frac{|d\alpha|}{2\pi}, \tag{3.4}$$

where the above integral is understood in the weak sense, i.e.,

$$\langle A_\varphi f, g\rangle = \int_{\mathbb{T}} \langle \varphi(U_\alpha) f, g\rangle \frac{|d\alpha|}{2\pi}, \quad f, g \in K_\vartheta.$$

When $\varphi \in L^2$ (not necessarily bounded), there is also a version of this formula, although it must be interpreted in a very special way. Sarason also proves that $\mathcal{T}_\vartheta$ is closed in the weak operator topology. Is it the case that

$$\mathcal{T}_\vartheta := \bigvee \{q(U_\alpha) : q \text{ is a trigonometric polynomial}; \alpha \in \mathbb{T}\}? \tag{3.5}$$

In the above, $\bigvee$ is the closed linear span in the weak operator topology. This is certainly true when ϑ is a finite Blaschke product (Theorem 1.18). In order to prove (3.5), it suffices, by means of (3.4), to prove that $\{A_\varphi : \varphi \in L^\infty\}$ is dense (weak operator topology) in $\mathcal{T}_\vartheta$. As mentioned earlier, it is unknown whether or not the above set is actually equal to $\mathcal{T}_\vartheta$.

4. Proof of Theorem 1.11 and Theorem 1.15

Fix $\alpha_1, \alpha_2 \in \mathbb{T}$ with $\alpha_1 \neq \alpha_2$. Let $\{\zeta_j, \eta_j : j = 1, \cdots, n\}$ be the points of $\mathbb{T}$ for which

$$B(\zeta_j) = \beta_{\alpha_1}, \quad B(\eta_j) = \beta_{\alpha_2}, \quad j = 1, \cdots, n.$$

We know from Remark 3.3 that any member of $\mathcal{T}_B$ takes the form

$$\sum_{j=1}^{n} c_j k_{\zeta_j} \otimes k_{\zeta_j} + \sum_{j=1}^{n} d_j k_{\eta_j} \otimes k_{\eta_j}$$

for some complex constants c_j, d_j. Let $e_{\zeta_s} = w_s v_{\zeta_s}$, where

$$w_s = e^{-\frac{i}{2}(\arg(\zeta_s) - \arg(\beta_{\alpha_1}))}. \tag{4.1}$$

To prove Theorem 1.15, let us compute

$$\left\langle \left(\sum_{j=1}^{n} c_j k_{\zeta_j} \otimes k_{\zeta_j} + \sum_{j=1}^{n} d_j k_{\eta_j} \otimes k_{\eta_j} \right) e_{\zeta_p}, e_{\zeta_s} \right\rangle,$$

the matrix representation of this operator with respect to the basis $\{e_{\zeta_1}, \cdots, e_{\zeta_n}\}$

Since $\{e_{\zeta_1}, \cdots, e_{\zeta_n}\}$ is an orthonormal basis for K_B, every $f \in K_B$ has the 'Fourier' expansion

$$f(z) = \sum_{s=1}^{n} \langle f, e_{\zeta_s}\rangle e_{\zeta_s}(z) = \sum_{s=1}^{n} \frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}} f(\zeta_s) e_{\zeta_s}(z)$$

and so

$$\langle f, g\rangle = \sum_{s=1}^{n} \frac{f(\zeta_s)\overline{g(\zeta_s)}}{|B'(\zeta_s)|}, \quad f, g \in K_B. \tag{4.2}$$

First notice that

$$e_{\zeta_s}(\zeta_q) = \frac{w_s}{\sqrt{|B'(\zeta_s)|}} k_{\zeta_s}(\zeta_q) = \begin{cases} w_s \sqrt{|B'(\zeta_s)|}, & \text{if } s = q; \\ 0, & \text{if } s \neq q. \end{cases}$$

From the above inner product formula in (4.2), we have

$$\begin{aligned}
\big\langle (k_{\zeta_j} \otimes k_{\zeta_j}) e_{\zeta_p}, e_{\zeta_s} \big\rangle &= \sum_{q=1}^{n} \frac{((k_{\zeta_j} \otimes k_{\zeta_j}) e_{\zeta_p})(\zeta_q) \overline{e_{\zeta_s}(\zeta_q)}}{|B'(\zeta_q)|} \\
&= ((k_{\zeta_j} \otimes k_{\zeta_j}) e_{\zeta_p})(\zeta_s) \frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}} \\
&= \frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}} \langle e_{\zeta_p}, k_{\zeta_j}\rangle k_{\zeta_j}(\zeta_s) \\
&= \frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}} \frac{w_p}{\sqrt{|B'(\zeta_p)|}} k_{\zeta_p}(\zeta_j) k_{\zeta_j}(\zeta_s) \\
&= \begin{cases} |B'(\zeta_s)|, & \text{if } s = p = j; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

In a similar way,

$$\begin{aligned}
\langle (k_{\eta_j} \otimes k_{\eta_j}) e_{\zeta_p}, e_{\zeta_s}\rangle &= \frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}} \frac{w_p}{\sqrt{|B'(\zeta_p)|}} k_{\zeta_p}(\eta_j) k_{\eta_j}(\zeta_s) \\
&= \frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}} \frac{w_p}{\sqrt{|B'(\zeta_p)|}} \frac{1 - \overline{B(\zeta_p)}B(\eta_j)}{1 - \overline{\zeta_p}\eta_j} \frac{1 - \overline{B(\eta_j)}B(\zeta_s)}{1 - \overline{\eta_j}\zeta_s} \\
&= \frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}} \frac{w_p}{\sqrt{|B'(\zeta_p)|}} \frac{1 - \overline{\beta_{\alpha_1}}\beta_{\alpha_2}}{1 - \overline{\zeta_p}\eta_j} \frac{1 - \overline{\beta_{\alpha_2}}\beta_{\alpha_1}}{1 - \overline{\eta_j}\zeta_s} \\
&= |1 - \overline{\beta_{\alpha_2}}\beta_{\alpha_1}|^2 \frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}} \frac{w_p}{\sqrt{|B'(\zeta_p)|}} \frac{1}{1 - \overline{\zeta_p}\eta_j} \frac{1}{1 - \overline{\eta_j}\zeta_s} \\
&= |1 - \overline{\beta_{\alpha_2}}\beta_{\alpha_1}|^2 \frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}} \frac{w_p}{\sqrt{|B'(\zeta_p)|}} \frac{1}{1 - \overline{\zeta_p}\eta_j} \frac{1}{1 - \overline{\eta_j}\zeta_s} \frac{(-\eta_j)\zeta_p}{(-\eta_j)\zeta_p} \\
&= -\eta_j |1 - \overline{\beta_{\alpha_2}}\beta_{\alpha_1}|^2 \frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}} \frac{w_p \zeta_p}{\sqrt{|B'(\zeta_p)|}} \frac{1}{\eta_j - \zeta_s} \frac{1}{\eta_j - \zeta_p}
\end{aligned}$$

The definition of w_p from (4.1) yields the identity

$$\zeta_p = \beta_{\alpha_1} \overline{w_p}^2.$$

Use this identity to manipulate the last line of the above expression to

$$-\beta_{\alpha_1}\eta_j |1 - \overline{\beta_{\alpha_2}}\beta_{\alpha_1}|^2 \frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}} \frac{\overline{w_p}}{\sqrt{|B'(\zeta_p)|}} \frac{1}{\eta_j - \zeta_s} \frac{1}{\eta_j - \zeta_p}.$$

Putting this all together, we get

$$\left\langle \left(\sum_{j=1}^{n} c_j k_{\zeta_j} \otimes k_{\zeta_j} + \sum_{j=1}^{n} d_j k_{\eta_j} \otimes k_{\eta_j} \right) e_{\zeta_p}, e_{\zeta_s} \right\rangle$$

$$= c_p |B'(\zeta_p)| \delta_{s,p} - \beta_{\alpha_1} |1 - \overline{\beta_{\alpha_2}} \beta_{\alpha_1}|^2 \sum_{j=1}^{n} \eta_j d_j \frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}} \frac{\overline{w_p}}{\sqrt{|B'(\zeta_p)|}} \frac{1}{\eta_j - \zeta_s} \frac{1}{\eta_j - \zeta_p}.$$

Using the partial fraction decomposition

$$\frac{1}{\eta_j - \zeta_s} \frac{1}{\eta_j - \zeta_p} = \frac{1}{\zeta_s - \zeta_p} \left(\frac{1}{\eta_j - \zeta_s} - \frac{1}{\eta_j - \zeta_p} \right),$$

one can verify the identities in (1.16). Thus the matrix representation of any truncated Toeplitz operator - with respect to the basis $\{e_{\zeta_1}, \cdots, e_{\zeta_n}\}$ - satisfies the conditions in (1.16). The proof of the converse is nearly the same as the proof of the converse in Theorem 1.4.

Using similar calculations as in the proof of Theorem 1.15, one proves that

$$\left\langle \left(\sum_{j=1}^{n} c_j k_{\zeta_j} \otimes k_{\zeta_j} + \sum_{j=1}^{n} d_j k_{\eta_j} \otimes k_{\eta_j} \right) v_{\zeta_p}, v_{\zeta_s} \right\rangle$$

$$= c_p |B'(\zeta_p)| \delta_{s,p} - \beta_{\alpha_1} |1 - \overline{\beta_{\alpha_2}} \beta_{\alpha_1}|^2 \sum_{j=1}^{n} \eta_j d_j \frac{1}{\sqrt{|B'(\zeta_s)|}} \frac{\zeta_p}{\sqrt{|B'(\zeta_p)|}} \frac{1}{\eta_j - \zeta_s} \frac{1}{\eta_j - \zeta_p}.$$

Now follow the rest of the proof of Theorem 1.15 to prove Theorem 1.11.

References

1. J. A. Cima, A. L. Matheson, and W. T. Ross, *The Cauchy transform*, Mathematical Surveys and Monographs, vol. 125, American Mathematical Society, Providence, RI, 2006.
2. D. N. Clark, *One dimensional perturbations of restricted shifts*, J. Analyse Math. **25** (1972), 169–191.
3. P. L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.
4. S. R. Garcia and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1285–1315 (electronic).
5. D. Sarason, *Sub-Hardy Hilbert spaces in the unit disk*, University of Arkansas Lecture Notes in the Mathematical Sciences, 10, John Wiley & Sons Inc., New York, 1994.
6. ______, *Algebraic properties of truncated Toeplitz operators*, Operators and Matrices **1** (2007), no. 4, 491–526.

Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27599

E-mail address: cima@email.unc.edu

Department of Mathematics and Computer Science, University of Richmond, Richmond, Virginia 23173

E-mail address: wross@richmond.edu

Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27599

E-mail address: wrw@email.unc.edu