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The logico-mathematical philosophy of Bertrand Arthur Russell

Elie Maynard Adams

THE LOGICO-MATHEMATICAL PHILOSOPHY

OF

BERTRAND ARTHUR RUSSELL

BY

ELIE MAYNARD ADAMS

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Chapter I

Statement of the Problem

Our purpose in this study is to give a critical exposition of Bertrand Arthur Russell's logico-mathematical philosophy. The thesis, which is the core of all his work on this subject, is the contention that logic and pure mathematics form a continuous whole, or as Russell himself states it: "They differ as boy and man: logic is the youth of mathematics and mathematics is the manhood of logic."[1]

In order to demonstrate the validity of his thesis, it was necessary for Russell to define what he meant by pure mathematics, to develop his logical calculus, and to demonstrate that all pure mathematics (including geometry) could be deduced from the principles of his logic. It is necessary for us to consider each of these subjects.

The primary problem with which we are concerned in this study is to determine whether or not Russell's proof, or demonstration, of his thesis is valid. If we determine that his proof is valid, then it follows that his thesis is valid also, but, if we determine that his proof is invalid, it does not necessarily follow that the thesis is invalid also. If we reach the latter conclusion, we can

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1. Introduction to Mathematical Philosophy (1919), p. 194.

only say that Russell has not proved his thesis, but this does not mean that the thesis itself is not true. For this reason we say that our problem is an examination of his proof and not of his thesis. Perchance we find that his proof is valid, then we will conclude that the thesis is valid also.

In addition to these matters related directly to the establishment of his thesis, we shall consider the history of the development of his thesis and the philosophical importance of mathematical logic.

Thus our study consists of four parts: (1). the history of the development of Russell's logico-mathematical thesis, (2). Russell's establishment of his thesis, (3). examination of his proof of his thesis, and (4). the philosophical importance of logico-mathematical philosophy.

Chapter II

Russell's Logico-Mathematical Thesis and Its History

The intellectual beginnings of Bertrand Arthur Russell (1872 -- ___) were in logic and mathematics and it is in this feld that he has done his most significant work and has won for himself a worthy name among the immortals of all ages. His book, done in collaboration with Alfred N. Whitehead, Principia Mathematica, which represents his mature work in mathematical logic, is listed among the one hundred greatest books of civilization.[1] Russell's work in this field began very early, being influenced chiefly by Bradley's logic until about 1898, when he was forced to change his opinions due to arguments on philosophy with G. E. Moore. At the age of twenty-four years, he published his first article which was entitled "The Logic of Geometry."[2] It was not until the publication of the article "Recent Work on the Principles of Mathematics" in the International Monthly (1901) that the influence of Peano upon him was revealed. From the time of his first article in 1896 until the publication of Introduction to Mathematical Philosophy (1919), which he wrote

1. St. John's College, Annapolis, Md., has a unique curriculum in which the four years work for the B. A. degree consists of mastering the one hundred greatest books of civilization. Principia Mathematica is included in this list. See Life Magazine, Feb. 5, 1940, pp. 61-67.
2. Mind, n.s., vol. 5 (1896), pp. 1-23. This is Russell's first article as far as I can determine by Poole's Index to Periodical Literature.

during his four and one half months imprisonment in 1918, his chief contributions were in the field of mathematical logic. During this period there appeared, in addition to numerous articles on the subject, the following books: (1). *An Essay on the Foundations of Geometry* (1897); (2). *A Critical Exposition of the Philosophy of Leibniz* (1900), which might have been entitled *A Study of Leibniz's Logic and Mathematics* without doing any injustice worthy of note to its contents; (3). *The Principles of Mathematics* (1903); in collaboration with Whitehead, *Principia Mathematica*, vol. 1 (1910), vol. 2 (1912), vol. 3 (1913); and (4). *Introduction to Mathematical Philosophy* (1919).

The primary thesis of ~~all~~ these logico-mathematical works is that pure mathematics and logic are identical, or as he states in the preface to the 1903 edition of *Principles of Mathematics*, his objective in this book (and it is the same in all of them) is "the proof that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles." This thesis with the ~~firm~~ demonstration of it in collaboration with Whitehead in *Principia Mathematica*, by strict symbolic reasoning with "all the certainty and precision of which mathematical demonstrations are capable," is Russell's greatest achievement.

In proving (or should we say demonstrating?) this thesis, a new science has been perfected, which is known as "mathematical" (or symbolic) logic. This science is defined for us by Russell himself in this way: "By the name 'mathematical logic'...I will denote any logical theory whose object is the analysis and deduction of arithmetic and geometry by means of concepts which belong evidently to logic."[1]

Although we have called this a new science, let us hasten to add that work had been done previously on the subject. The idea that logic and mathematics come from the same roots must have been felt even by Euclid and his disciples, since what they called "Common Notions" are formulations of logical principles, and the characteristics of space are described in the definitions and postulates. But it was not until the time of Leibniz that the significance of this kinship was actually felt. The history of the subject properly begins with him.[2] Leibniz has Philalethes, in the New Essays on the Human Understanding, to say: "I begin to form for myself a wholly different idea of logic from that which I formerly had. I regarded it as a scholar's diversion, but I now see that, in the way you understand it, it is like a universal mathematics."[3] In this statement we see that he had caught a glimpse of the larger outlines of the subject, but he never understood the difficulties involved and he contributed little to the successful working out of the details. However, he did advocate two necessary features of the science. He contended for a u-

1. Russell, "The Philosophical Importance of Mathematical Logic," Monist, vol. 23 (1913), p. 481.
2. See C. I. Lewis, A Survey of Symbolic Logic (Berkeley, Cal.: University of California Press, 1918), p. 5.
3. Book IV, Chapter XVII, Paragraph 9; quoted in Ibid, p.5.

niversal medium for the expression of science and for a calculus of reasoning "designed to display the most universal relations of scientific concepts and to afford some systematic abridgment of the labor of rational investigation in all fields, much as mathematical formulae abridge the labor of dealing with quantity and number."[1] For the universal medium he recommended an ideographic language rather than phonographic. He wanted certain fundamental characters or symbols which would be the "alphabet of human thought," and these and combinations of them would form the symbolism of the new science. His hope has been largely realized in Peano, Russell and Whitehead, but Leibniz, partly due to his own nature and partly because he spent his time trying to win learned societies to the acceptance of his thesis and thus instigate a revolution in the methods of science in a short time, neglected the more limited task of working out the details and the technique necessary to prove his theory feasible.

Immanuel Kant felt very keenly the kinship between logic and mathematics. He may have been guided in this by his great contemporary, Lambert. But he was prevented from identifying logic and pure mathematics like Russell has done because of three factors. First, he thought of logical propositions as being analytic and mathematical propositions

1. C. I. Lewis, op. cit., p. 6.

as synthetic, but it has been proved that logic is just as synthetic as all other kinds of truth.[1] Second, in Kant's time formal (or symbolic) logic was in a very undeveloped stage in comparison to the work of the past century. Kant himself held that no great advance had been made in logic since Aristotle, and Aristotle did not go beyond the syllogism. This was certainly inadequate for mathematics. And, third, mathematical reasoning itself was very inferior in Kant's day to what it was when Russell began his work. Mathematics had not, at that time, been completely divorced from empiricism. Russell says, in a drastic statement, "there probably did not exist, in the eighteenth century, any single logically correct piece of mathematical reasoning, that is to say, any reasoning which correctly deduced its results from the explicit premises laid down by the author."[2]

Since the time of Kant, mathematicians and logicians alike have felt the need for reexamination of the nature of thought-operations and of extending concepts of logic in much the same way as metageometricians endeavored to construct a pangeometry which would be free from Euclidean space. The first great work to advance Aristotelian logic was The Laws of Thought (1854) by George Boole. Russell says that "pure mathematics was discovered by Boole...";[3] but Boole asserted time and again that his book was not on

1. See Russell, A Critical Exposition of the Philosophy of Leibniz (1900), pp. 16ff.
2. Principles of Mathematics (1938 edition, which will be used throughout this work, since it does not differ from the 1903 edition except for a new Introduction) p. 457.
3. "Recent Work on the Principles of Math." Int. Monthly, vol. 4 (1901), p.83.

mathematics. Russell says that this belief of Boole's was due to the fact that he was too modest to think that he was the first person who had ever written a book on pure mathematics. Since the publication of Boole's book (1854), which was a significant date in the history of logic because it marked the end of a period which had been dominated almost exclusively by Aristotle, logicians have realized the insufficiency of Aristotelian methods. They have seen the need of deciphering the nature of thought in its operations, and they have attempted "to exhibit the functions of reason in formulas, or in graphic presentations, or in algebraic notations."[1]

The chief workers in this field prior to Russell were Ernst Shroeder, Charles S. Peirce, Giuseppe Peano, and Frege. It is interesting that Frege developed theories very similar to those of Russell, but Russell arrived at his independently for the most part. Russell very explicitly recognizes his indebtedness to these, especially to Peano, whom he calls the great master of formal reasoning.[2] Much more progress was made in formal logic in each decade from 1850 to 1915 than in all the centuries from Aristotle to Boole. <u>Principia Mathematica</u> stands as the great capstone or the culmination of all the work that had been done on the subject in all of its past history.

1. P. Carus, "The Nature of Logical and Mathematical Thought," <u>Monist</u>, vol. 20 (1910), p. 43.
2. "Recent Work on the Principles of Mathematics," <u>Int. Monthly</u>, vol. 4 (1901), p. 86.

The great advance in the science of logic during the last half of the nineteenth century and the first two decades of the twentieth has been due to the invention of a stricter symbolism by Boole, Shroeder, Peirce, and especially Peano, Russell, and Whitehead. Two hundred years before Peano's work, Leibniz foresaw this new science and sought to create it, but he was prevented from doing so partly by his inability to believe Aristotle to be guilty of definite formal fallacies. He hoped, through the means of symbolic logic, for a solution of all philosophical problems, and thus an end of all disputes. "If controversies were to arise," he says, "there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pens in their hands, to sit down to their desks, and to say to each other (with a friend as witness, if they liked), 'Let us calculate.'"[1] This was excessive optimism. With all the strictness and precision of the symbolism of Russell and Whitehead, there still remain problems whose solutions are doubtful and disputes which calculation is unable to decide even within the realm of mathematical philosophy. But Leibniz's dream has become true for a wide field of what was previously controversal. Symbolic logic has extended the certainty of mathematics to much of mathematical philosophy, which was previously just as controversal as any other field of philosophy.

1. Quoted by Russell in "Recent Work on the Principles of Mathematics," Int. Monthly, vol. 4, p. 87.

Concerning the importance of symbolism, Russell says:

> It is not easy for the lay mind to realize the importance of symbolism in discussing the foundations of mathematics, and the explanation may perhaps seem paradoxical. The fact is that symbolism is useful because it makes things difficult. (This is not true of the advanced parts of mathematics, but only of the beginnings.) What we wish to know is, what can be deduced from what. Now, in the beginnings, everything is self-evident; and it is very hard to see whether one self-evident proposition follows from another or not. Obviousness is always the enemy of correctness. Hence we invent some new and difficult symbolism, in which nothing seems obvious. Then we set up certain rules for operating on the symbols, and the whole thing becomes mechanical. In this way we find out what must be taken as premise and what can be demonstrated or defined.[1]

In this statement, we see something of Russell's love of stating his ideas in paradoxical form. Yet his paradox holds true. But this statement has a greater significance than merely revealing our philosopher's love for paradoxes. It gives us in clear terms his method of determining which are the primitive ideas and propositions from which all else is deduced in logic and mathematics. This constitutes an importance second to none in his logico-mathematical philosophy.

Symbolism in logic had accomplished much before the work of Russell and Whitehead. In 1901, Russell wrote that due to symbolic logic "many of the topics which used to be placed among the great mysteries--for example, the natures of infinity, of continuity, of space, time, and

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1. "Recent Work on the Principles of Mathematics," _Int. Monthly_, vol. 4, pp. 85-86.

motion--are now no longer in any degree open to doubt or discussion. Those who wish to know the nature of these things need only read the works of such men as Peano and Georg Cantor..."[1]

Peano's idea of symbolism in logic was not new. It dates back to Leibniz (as we have already observed) and Descartes, and perhaps earlier, but it was not until about the middle of the nineteenth century that it began to be vigorously developed by the labors of Boole, De Morgan, and others. Peano became the great master of the subject and laid the foundations upon which Russell and Whitehead built. However, in some respects his work was not nearly so fundamental and subtle as Frege's, but his views and methods became far better known than the German's because of his editing and publishing a journal and a periodical collection of mathematical propositions expressed in his symbolism. Peano's symbolism largely consisted of certain convenient signs for denoting logical notions so that logical propositions could be translated into a form similar to mathematical equations. One significant result of Peano's work was the discovery that all the ideas which appear in arithmetic, geometry, and other mathematical sciences can be defined in terms of the ideas of general logic, such as class, implication, class inclusion, conjunction and disjunction of classes, together with several other ideas like

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1. "Recent Work on the Principles of Mathematics," Int. Monthly, vol. 4, p. 88.

integer, number, and point. Also Peano contributed much of the utmost importance to logic, such as the idea that inference in mathematics is not the inference of one proposition from another, but the inference of a whole class of propositions from another class.[1] But as significant as Peano's work was, he was prevented from developing it further by his failure to deal with the logic of relations, which had been founded and developed considerably by De Morgan, C. S. Peirce, and Schroeder.

Russell, partly helped by his study of Frege's work, and partly having discovered independently many of Frege's distinctions, took up Peano's work where Peano had left it and added, among many other things, the logic of relations. He defined in logical terms alone all of the fundamental mathematical propositions of Peano and developed and demonstrated the thesis that mathematics and logic form parts of a dontinuous whole. All of this he did with a minimum number of undefined ideas and undemonstrated propositions.

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1. P. E. B. Jourdain, "Some Modern Advances in Logic," Monist, vol. 21 (1911), pp. 564-565.

Chapter III

Definition of Pure Mathematics

Traditionally mathematics was supposed to have been concerned with questions about number and quantity. It was defined as the science of number and quantity, with quantity being defined as that of which we can predicate the relations of equality, greater than, or less than. But this definition has had to give way before the rise of distinctively non-quantitative mathematical sciences, like the theory of aggregates (or manifolds), projective geometry, and analysis.

Non-Euclidean geometry was the first of the mathematical sciences to free itself from empiricism. Euclideans assumed that geometry dealt with the space in which we live. But, as Russell says, "it has gradually appeared by the increase of non-Euclidean systems, that Geometry throws no more light upon the nature of space than arithmetic throws upon the population of the United States." He adds, "Geometry is a whole collection of deductive sciences based on a corresponding collection of sets of axioms. One set of axioms is Euclid's; other equally good sets of axioms lead to other results."[1]

Georg Cantor, to whom Russell is greatly indebted, completed his work on transfinite (or infinite) numbers and ordinal types in 1895 and 1897 in two epoch

1. "Recent Work on the Principles of Mathematics," Int. Monthly, vol. 4, p. 98.

making articles[1] in which the principles of the subject were stated in an almost perfect logical form. With the work of Peano and others, logic was becoming more like mathematics, that is it was being reduced to pure symbolic form, and with the work of the non-Euclidean geometricians and Cantor, mathematics was being reduced to logical forms. Thus by the symbolization of logic and the logicalization of mathematics, the way was paved for Russell's thesis that logic and pure mathematics are one, logic being the boyhood of mathematics and mathematics being the manhood of logic.

By 1901 mathematics had become generalized and divorced from the world of quantity and particulars to such an extent that Russell defined (perhaps "described" would be better) the subject in this manner: "Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."[2] This was a bombshell to the traditional mathematicians and philosophers for they had believed that pure mathematics was the one remaining field where agnosticism had no opportunity to establish itself. This bold statement was attacked on

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1. Mathematische Annalen, vols. XLVI and XLIX. Cf. Jourdain, "Transfinite Numbers and the Principles of Mathematics," Monist, January, 1910.
2. "Recent Work on the Principles of Mathematics," Int. Monthly, vol. 4 (1901), p. 84. Russell stated his fundamental views on mathematical logic in this article. These views were given further elaboration in The Principles of Mathematics (1903) and in subsequent works. We are told, however, in the preface to the second edition (1938) of The Principles of Mathematics that the work was written for the most part in 1900. Therefore, we may consider the 1901 article as an abbreviated edition of the fundamental views of The Principles of Mathematics.

all flanks, but while it was being fought over Russell and Whitehead were occupied with giving this thesis a detailed and precise demonstration in Principia Mathematica in such a fashion that no one has been able to refute it to date on their own grounds of demonstration. It is true that certain defects have been found, some of which have been corrected, but the work still stands as the masterpiece and authority on the subject. Although this definition seems to be agnostic in nature, and it is as far as knowledge of the external world through mathematics is concerned, it makes it possible for more complete knowledge with more certainty to be established in the field of mathematics.

As E. T. Bell has pointed out,[1] the definition of mathematics quoted above has four great merits: (1). It shocks the self-conceit out of common sense; (2). it emphasizes the completely abstract nature of mathematics; (3). it suggests the reduction of all mathematics and the more mature sciences to the postulational form so that all people, mathematicians, philosophers, scientists, and ordinary plain common sense, can see precisely what it is that each of them imagines that he is talking about, and (4). "Russell's description of mathematics administers a resounding parting salute to the doddering tradition, still respected by the makers of dictionaries, that mathematics is the science of number, quantity, and measurement." These

1. E. T. Bell, The Queen of the Sciences, (Baltimore: The Williams and Wilkins Company, 1931), pp. 16f.

things, according to the new mathematics, only constitute an important part of the materials to which mathematics has been applied, rather than being an essential part of mathematics itself.

Russell's definition that mathematics is "the subject in which we never know what we are talking about, nor whether what we are saying is true" needs clarification, and this we find in Russell's own words. He explains:

> We start, in pure mathematics, from certain rules of inference, by which we can infer that if one proposition is true, then so is some other proposition. These rules of inference constitute the principles of formal logic. We then take any hypothesis that seems assuring [or amusing],[1] and deduce its consequences. If our hypothesis is about anything, and not about some one or more particular things, then our deductions constitute mathematics... Now the fact is that, though there are indefinables and indemonstrables in every branch of applied mathematics, there are none in pure mathematics except such as belong to general logic. Logic, broadly speaking, is distinguished by the fact that its propositions can be put into a form in which they apply to anything whatever. All pure mathematics -- Arithmetic, Analysis, and Geometry --is built up by combinations of the primitive ideas of logic, and its propositions are deduced from the general axioms of logic, such as the syllogism and the other rules of inference.[2]

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1. Concerning this, Jourdain, in an article entitled "Mr. Bertrand Russell's First Work on the Principles of Mathematics," in the Monist, vol. 22(1912), p. 149, says that he learned from a copy of Russell's article which had been corrected by its author, that the typesetter or editor had substituted "assuring" for what was originally "amusing." Mr. Jourdain adds that this substitution "took away from the force of Mr. Russell's contention that in mathematics we are not in the least concerned with the truth or otherwise of our hypotheses or consequences, but merely with the truth of the deductions."
2. "Recent Work on the Principles of Mathematics," Int. Monthly, vol. 4 (1901), p. 84.

These remarks explicitly defines pure mathematics in accord with his fundamental thesis. They make clear the complete abstractness and generality of pure mathematics and that it is all deduced from the fundamental primitive principles of logic. Concerning these primitive principles, Russell says that "there are at most a dozen notions out of which all the notions of pure mathematics (including Geometry) are compounded"[1] and that "the whole of arithmetic and algebra has been shown to require three indefinables and five indemonstrable propositions."[2]

We now come to a more technical definition of pure mathematics. It is as follows:

> Pure mathematics is the class of all propositions of the form "p implies q," where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of such that, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics uses a notion which is not a constituent of the propositions which it considers, namely the notion of truth.[3]

At this time we shall not consider the meaning of some of the terms used in this definition, since they are to occupy a considerable amount of attention later. The important thing about this novel definition, at least it was novel for its time, is that it defines pure mathematics as

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1. "Recent Work on the Principles of Math.," Int. Monthly, vol. 4, p. 87.
2. Ibid, p. 86. This quote refers to the work of Peano.
3. Russell, The Principles of Mathematics, (1903), p. 2.

being concerned with implications, and not with statements in which their hypotheses are realized. Applied mathematics has to do with the latter. Boole and Peano made class and class-inclusion primary in symbolic logic, but Russell, as we learn from the above definition, makes propositions (as distinct from propositional functions, which are to be explained later), and material implication (which is distinguished, as we shall show later, from formal implication, which has to do with propositional functions) as primary. This is quite significant. He demonstrates how class-inclusion may always be expressed as implication. For example, "all men are mortal" may be stated as "x is a man implies x is a mortal," which means that any man is included in the class of mortals.

This definition of pure mathematics is the culmination of the discovery contributed to by Leibniz, Frege, Dedekind, Schroeder, Peano and many others. But it is only the beginning in Russell's work. He revises it considerably as he progresses in his thought on the subject through the years. Although it will involve presupposition of much that is to come later in our discussion, let us at this point consider some of the changes which Russell has made in the above definition. If all that follows is not clear at this time, it is probably due to presupposition of material not yet explained, and it is suggested to the reader that he reread this portion after having completed our discussion of Russell's logic.

We are fortunate in having Russell's own criticism and revision of this definition in the Introduction to the second edition of The Principles of Mathematics (1938). This criticism and revision comes after more than forty years work in the field, and we may take it as probably his final treatment of the definition of pure mathematics. He makes three revisions in the definition. First, concerning the form "p implies q," he says that it is "only one of many logical forms that mathematical propositions may take."[1] In his original definition of 1903, he had thought that it was the only form. He attributes this "undue stress on implication, which is only one among truth-functions, and no more important than the others"[2] to his consideration of geometry in his early works. In order to include Euclidean and non-Euclidean systems alike in pure mathematics without regarding them as mutually inconsistent, it was necessary to assert only that the axioms imply the propositions, and not that the axioms are true and therefore the propositions are true. This and other instances brought implication into prominence and made it appear to be fundamental. He still regards it possible to define mathematical propositions in this way, but it is now considered as an arbitrary selection of several possible ways instead of its being regarded as the only way as he did formerly.

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1. Principles, p. vii. From this point on we shall refer to Principles of Mathematics merely as Principles.
2. Ibid, p. vii.

The second change concerns the statement "p and q are propositions containing one or more variables." He agrees with many of his critics that it would be more correct to say that they are propositional functions rather than propositions. This error was pointed out by A. T. Shearman in his review of the first edition of the book.[1] But Russell excuses his error on the ground that propositional functions had not at that time been defined. However, he defined the term in the second chapter of the book, and, therefore, he should have introduced the idea at this point even if he had postponed its definition to its rightful place in the development of the book. The remainder of the book is given to justifying the definition and surely the definition should have been consistent with the material presented as the justification of it. The error was clearly due to an oversight or to the lack of thorough comprehension of all the distinctions he was to make or had made in the body of the book.

The third change deals with a more serious matter, namely, the statement that "neither p nor q contains any constants except logical constants." Assuming that we know what logical constants are, which will be explained later, we may point out the weakness of this statement. Russell says in 1938 that "the absence of non-logical constants, though a necessary condition for the mathematical character of a proposition, is not a sufficient condition."[2] In other

1. Mind, n.s., vol. 16, p. 254.
2. Principles, p. vii.

words, a mathematical proposition must not have any non-logical constants, but a proposition is not mathematical merely by the virtue of not having any non-logical constants. Thus a proposition may satisfy the definition in the Principles of Mathematics and not be capable of logical or mathematical proof or disproof. The definition includes all mathematical propositions, but it does not exclude all non-mathematical propositions. An example, as Russell points out, of a statement that is included in the definition but is not mathematical is any statement concerning the number of things in the world. For example; "There are at least three things in the world," which is equivalent to "there exist objects x, y, z and properties φ, ψ, χ such that x but not z has the property ψ, and y but not z has the property χ."[1] This statement can be expressed in purely logical terms, and it can be logically proved to be true of classes of classes of classes. There must be at least four classes of classes, even if the universe were non-existent. In that case there would be the null-class; two classes of classes, namely, the class of no classes, and the class of which only the null-class is a member; and four classes of classes in all, namely, the null-class, the one of which only the null-class of classes is a member, the one of which its only member is the class whose only member is the null-class, and the class which is the sum of the last two. But among the lower types,

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1. Principles, pp. vii-viii.

such as that of individuals, of classes, and of classes of classes, it cannot logically be proved that there are at least three members. This presupposes knowledge of the theory of types, which appeared in a crude form in Appendix B of the first edition of The Principles of Mathematics, but it was not until the publication of an article entitled "Mathematical Logic as based on the Theory of Types" in the American Journal of Mathematics in 1908 that the thoery was well developed. It is fundamental in much of Principia Mathematica.

Another example cited by Russell[1] is the multiplicative axiom or Zermelo's axiom of selection, which is its equivalent. We shall discuss this axiom later, and, consequently, only a few words will suffice at this time. The axiom asserts "that, given a set of mutually exclusive classes, none of which is null, there is at least one class consisting of one representative from each class of the set."[2] No one knows the truth or falsity of this statement. One can easily imagine universes in which this is true, but it is an impossibility to prove that there are possible universes in which it would be false. Also it is an impossibility to prove that there are no possible universes in which it would be false.

Therefore, since the definition of pure mathematics given in the beginning of the Principles includes all mathematical propositions, but it does not exclude all

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1. Principles, p. viii.
2. Ibid, p. viii.

non-mathematical propositions, it is necessary to add something else to the definition, if it is to define pure mathematics. "In order that a proposition may belong to mathematics," Russell says in 1938, "it must have a further property: according to Wittgenstein it must be 'tautological,' and according to Carnap it must be 'analytic.'"[1] Whether a proposition is or is not analytic depends upon the premisses with which we begin. Consequently, the question as to what are logical propositions is largely arbitrary unless we have a criterion of admissable logical premisses.

The question of logical constants, which is quite important in the definition at the beginning of the Principles, goes through repeated modifications in the process of the development of Russell's thought on the matter. These modifications are to be treated else-where, but let it suffice at the present for us to quote a statement from the Introduction to the 1938 edition of the Principles. He says:

> Logical constants..., if we are to be able to say anything definite about them, must be treated as part of the language, not as part of what the language speaks about. In this way, logic becomes much more linguistic than I believed it to be at the time when I wrote the Principles. It will be true that no constants except logical constants occur in the verbal or symbolic expressions of logical propositions, but it will not be true that these logical constants are names of objects, as Socrates is intended to be.[2]

In 1938, Russell concludes that "to define logic, or mathematics, is...by no means easy except in relation to

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1. Principles, p. ix.
2. Ibid, p. xi-xii.

some given set of premisses."[1] He says that a logical premise must have two characteristics, namely, "complete generality, in the sense that it mentions no particular thing or quality," and "it must be true by virtue of its form."[2] We can define logic in relation to a definite set of logical premisses as "whatever they enable us to demonstrate." But this means a definition presents two great difficulties. First, it seems impossible to prove that a system resulting from a certain set of premisses includes everything that we should include among logical propositions, and second, what is meant by saying that a proposition is true in virtue of its form? Russell is forced to confess "I am unable to give any clear account of what is meant by saying that a proposition is true in virtue of its form."[3] But he believes that this phrase --"true in virtue of its form"-- points to the problem which must be solved if an adequate definition of logic is to be given.

Hence it is evident that Russell is less confident of his definition of pure mathematics thirty-five years later than he was at the time he published the Principles. This is evidence, however, that his mind has always been alert and continually grappling with old and new problems. This aspect of his nature has prevented him from believing in the finality of his thought and, consequently, it has prevented him from being dogmatic.

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1. Principles, p. xii.
2. Ibid, p. xii.
3. Ibid, p. xii.

Chapter IV

Russell's Logical Calculus

Russell uses the terms Symbolic, Formal, and Mathematical Logic as synonyms. Following the most common usage in England and America,[1] for the most part we shall designate the science under consideration by the name of symbolic logic. All logic, including the syllogism of Aristotle, is symbolic, but the name as we use it denotes that type of logic which is distinguished from various special branches of mathematics chiefly by its generality. The subject is defined by Russell as "the study of the various general types of deduction."[2] From this definition one would conclude that Russell excludes from symbolic logic the principle of induction which was discovered by Bacon and has become the basis of the scientific method. But Russell includes induction in deduction. He says: "I do not distinguish between inference and deduction. What is called induction appears to me to be either disguised deduction or a mere method of making plausible guesses."[3] Symbolic logic investigates "the general rules by which inferences are made, and it requires a classification of relations or propositions only in so far as these general rules introduce particular notions."[4] These particular

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1. C. I. Lewis, <u>op. cit.</u>, p. 1
2. <u>Principles</u>, p. 10.
3. <u>Ibid</u>, p. 11, f. n.
4. <u>Ibid</u>, p. 11.

notions in the propositions of symbolic logic, and all other notions definable in terms of these, are the logical constants, and they are the only constants which a proposition of pure mathematics can contain, since generality is the chief characteristic of such a proposition.

Logical Constants Defined

A constant is something which is absolutely definite, like Socrates, past, present, future, etc. A proposition cannot be characterized by complete generality and contain any such particulars. However, even pure mathematical propositions contain logical, but only logical, constants. It is in this respect that pure mathematics is distinguished from applied mathematics since the latter contains non-logical constants like definite objects.

Any constant is something absolutely definite, about which there can be no ambiguity. But what are logical constants? We have said above that they are particular notions, and all other notions definable in terms of these, which appear in the propositions of symbolic logic. In the definition of pure mathematics given at the beginning of the Principles, we are told that "logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of such that, the notion of relation, and such further notions as may be involved in the general notion of propositions" of pure mathematics. From the statement given above, namely, that the logical constants are the particular notions, and the notions definable in terms of these particular notions, introduced by the general rules of inference, it would seem that implication, the relation of a term to a class of which it is a member, the notion of such that, the notion of relation, etc. would be logical constants as well as the notions definable in terms of these. Although the definition of logical constants included in the definition of pure mathematics, as quoted above, does not say that these are logical constants, it is assumed that they are throughout the book. Here again we find a lack of consistency, or explictness, between the statement of the same idea in two places in less than a dozen pages apart. This must have been due to the fact that he was working his ideas out and defining his terms as he labored on the book.

It is in the Introduction to the second edition of the Principles (1938) that we find Russell's best treatment of the meaning of logical constants and the history of his thought on the matter. At the time that he wrote the Principles, Russell was Platonic in his agreement with Frege that logical constants in some way have constituents corresponding to their symbolic expressions. Such constants are the principles of disjunction (or), conjunction (and), negation (not), if-then, the null-class, 0, 1, 2, 3, etc. Even at the beginning in 1903 Russell did not believe in the reality of the celestial archtypes of disjunction, conjunction, negation, proposition, implication, such that, if-then, but he did believe that numbers, which are logical constants definable in terms of the indefinable logical constants, people the timeless realm of Being. But concerning the indefinables, he believed in 1903 that they have some unique meaning. He said: "every word occurring in a sentence must have some meaning."[1] By this he meant that it must have an intelligible use. This, however, is not true, as he later recognizes, when the word is taken in isolation. What is true is that every word in a sentence contributes to the meaning of the sentence, but that is quite different from the idea that each word has a meaning all its own.

In the development of his thought, Russell has been forced to give up his Platonic conception of the meaning of things. The first step in this process was the acceptance of the theory of descriptions. According to this theory, in the proposition "Russell is the author of the Principles of Mathematics," there is no constituent corresponding to "the author of the Principles of Mathematics." The analysis of this gives us the propositional function: "'x wrote the Principles of Mathematics is equivalent to x is Russell'-is true for all values of x." This theory does away with the contention that there must be in the realm of Being such objects as the square circle and the golden mountain since we can talk about such objects. The question of the square circle, and those similar to

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1. Principles, p. 42.

it, have always created difficulties because one naturally asks, when it is stated that the square circle does not exist, what is it that does not exist? The square circle was believed to have some kind of existence or meaning. But the theory of descriptions avoids this difficulty.

The next step, which caused Russell to give up even his Platonic conception of the existence of real numbers in the realm of Being, was the abolition of classes. This step was taken in the Principia Mathematica,[1] where it is said: "The symbols for classes, like those for descriptions, are, in our system, incomplete symbols; their *uses* are defined, but they themselves are not assumed to mean anything at all...Thus classes, so far as we introduce them, are merely symbolic or linguistic conveniences, not genuine objects." He had previously defined cardinal numbers as classes of classes similar to a given class, and, with the new step in Principia Mathematica[2], cardinal numbers became "merely symbolic or linguistic conveniences."

At this stage of his development, Russell says that[3] Whitehead persuaded him to substitute for points of space, instants of time, and particles of matter logical constructions composed of events. He adds: "In the end, it seems to result that none of the raw material of the world has smooth logical properties, but that whatever appears to have such properties is constructed artificially in order to have them."[3]

1. Vol. 1., pp. 71-72.
2. From this point on Principia Mathematica will be denoted merely by Principia.
3. Principles, p. xi.

As a result of these developments, Russell concludes that "Logical constants..., if we are to be able to say anything definite about them, must be treated as part of the language, not as part of what the language speaks about. In this way, logic becomes much more linguistic than I believed it to be at the time when I wrote the *Principles*."[1]

Since no proposition of logic can mention any particular object, the only constants in propositions of pure mathematics (or symbolic logic) are logical constants. In the *Principles*, Russell had more difficulty with this than he has had since he came to think of logical constants as "merely symbolic or linguistic conveniences," rather than as particular objects in some sense.

We have given considerable attention to Russell's conception of logical constants, and this has not been without a purpose, for he says that "pure mathematics must contain no indefinables except logical constants..."[2] Thus all of the primitive (indefinable) ideas from which all of mathematics is deduced are logical constants. They are the foundations, along with the indemonstrable (or rather the undemonstrated) propositions, upon which the whole of the science of mathematics is built. Therefore, it is necessary for us to have a clear understanding of what Russell means by them.

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1. *Principles*, p. xi.
2. *Ibid*, p. 8.

The Indefinable Logical Constants of the Principles

Before we deal with the indefinable logical constants, it is necessary for us to say something about what Russell means by "definability" or "definition." Philosophically, the word "definition" has been largely restricted to the analysis of an idea into its constituent parts, but Russell rejects this meaning of the term on the ground that wholes are not determinate when their parts are known.[1] He employs definitions only in a nominal sense. The definability of a term for him is always relative to a given set of notions. He says: "Given any set of notions, a term is definable by means of these notions when, and only when, it is the only term having to certain of these notions a certain relation which itself is one of the said notions."[2] Thus a term is definable only when it can be defined in terms of given notions. According to this method, there must always be at least one or more indefianble ideas by means of which other ideas are defined. These given undefined ideas are known as primitive.

In the Principles,[3] Russell says that the number of indefinable logical constants are eight or nine, but he proceeds to give several lists, varying in number and in the terms included. We should note that he agrees with Peano that notions which are taken as indefinables are to a certain extent arbitrary, but he says that "it is important

1. Principles, pp. 111-112.
2. Ibid, p. 111.
3. Ibid, p. 11.

to establish all the mutual relations of the simpler notions of logic, and to examine the consequences of taking various notions as indefinable."[1] In one place,[2] he lists six indefinables: (1). formal implication, (2). material implication ("implication between propositions not containing variables"), (3). the relation of a term to a class of which it is a member (which is, following Peano, denoted by ∈), (4). the notion of <u>such that</u>, and (5). the notion of truth. Then he states that "By means of these notions, all the propositions of symbolic logic can be stated." But on page 106, which is a summary of the section dealing with the "Indefinables of Mathematics," we have the following list: (1). implication, (2). the relation of a term to the class of which it is a member, (3). the notion of <u>such that</u>, (4). the notion of relation, (5). propositional function, (6). class, (7). denoting, and (8). <u>any</u> or <u>every</u> term.

The first list, as we observed, proposes to be complete, and, yet, the second list contains five new indefinables, having left out entirely the notion of formal implication (implication meaning material implication) and the notion of truth which were included in the first list. This difference, however, is not as grave as it appears. On further examination he discovered that formal implication is a complex concept and that it involves the notions of propositional function, class, denoting, and <u>any</u> or <u>every</u>

1. <u>Principles</u>, p. 27.
2. <u>Ibid</u>, p. 11.

term, the additional indefinables. However, the reason for leaving out the notion of truth in the second list is not known.

Taking the last list, with some modifications, as what he intended to present as the indefinable notions of pure mathematical logic (or pure mathematics), let us examine each of them briefly in order to see what he means by them. Although no definition is given of any of them, each is discussed and described. The reason that they are called indefinables is that, in order to define them, the term itself is believed to be used or presupposed, and this does not constitute a definition.

But before we discuss the indefinables, we should point out that they are divided among the three calculuses of the logic, namely, the propositional calculus, the calculus of classes, and the calculus of relations. Peano held that the calculus of classes is primary, but Russell develops his system upon the assumption that the propositional calculus is fundamental, and, consequently, we have implication as the first of the indefinables, since it is the primary factor in this calculus. In one place,[1] Russell says that the calculus of propositions requires two indefinables, namely, implication, meaning both material and formal implication. However, he also holds that the analysis of formal implication belongs to the subject, but is not required for its formal development,[2] and he ends

1. Principles, p. 15.
2. Ibid, p. 11

up by not including formal implication as an indefinable.[1] The calculus of classes requires three indefinables, namely, the relation of an individual, otherwise known as a term, to the class of which it is a member, the notion of a propositional function, and the notion of such that. The idea of a propositional function is employed in the calculus of propositions, but it is explained as it is used, and, therefore, the general notion is not necessary until we arrive at the second calculus. And the last calculus, the calculus of relations, requires only one indefinable, and that is the notion of relation itself. The other indefinables in the list are derived from the analysis of the complex notion of formal implication. We might mention that the notion of a propositional function, which is required in the calculus of classes, is also derived from this analysis.

We now proceed to a brief discussion of each of the indefinables. We must warn the reader not to take these as final in Russell's system of logic, but only as the terms which he had not been able to define in 1903. Before he had finished his most important works in the field, he had reduced the indefinables to two, as we shall observe as we trace his development.

1. Implication. Two kinds of implication are regarded as essential to every kind of deduction (and as we noticed previously, Russell included what is commonly re-

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1. Principles, p. 106.

garded as induction in his conception of deduction). These two kinds are simply implication, by which Russell always means material implication, and formal implication. But only the latter is regarded as indefinable in his final opinion in the Principles. The latter is considered as a complex notion, which involves the indefinables of propositional function, class, denoting, and any or every term. Implication has to do with propositions as distinguished from propositional functions, the latter of which is yet to be explained. It is "the relation in virtue of which it is possible for us validly to infer" between propositions.[1] This relation exists whether or not we can perceive it. Russell says that "the mind...is as purely receptive in inference as common sense supposes it to be in perception of sensible objects."[2] This idea of the reality of relations is very important in Russell's philosophy. Getting back to the relation between propositions which is known as implication, Russell says that "the relation holds...., when it does hold, without any reference to the truth or falsehood of the propositions involved."[3] Consequently, we get such absurd sounding statements as "any false proposition implies every proposition and any true proposition is implied by every proposition."[4]

It is impossible to define implication, Russell

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1. Principles, p. 33.
2. Ibid, p. 33.
3. Ibid, p. 33.
4. Ibid, p. 18. See f.n.

says, because any definition of it involves a vicious circle. For example, if we try to define it as meaning that "if one proposition is true, then another is true," the terms if and then already involve implication.

At this point, observe that Russell has in a way defined implication in saying that implication "is the relation in virtue of which it is possible for us validly to infer" one proposition from another. This does involve circularity in a sense in that "infer" is almost synonymous with "imply." But it seems that he is using the term "relation" as more fundamental than that of implication, and yet he asserts that implication must be regarded as the first and most fundamental of the indefinables. Nevertheless, we must admit, in all fairness to Russell, that he did admit in the beginning that the selection of indefinables is to a large extent arbitrary. But it seems apparent that implication can be defined as one of the many relations existing between propositions.

Although, in the last analysis, Russell does not regard formal implication as an indefinable, let us say something about it at this point in order to make a clear distinction between it and implication (material). He says that "a formal implication...is the affirmation of every material implication of...the class of all propositions in which a given fixed assertion, made concerning a certain subject or subjects, is affirmed to imply another given fixed assertion concerning the same subject or subjects."[1]

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1. Principles, p. 41.

In the above definition of formal implication, it is necessary to understand what is meant by the "assertion" of a proposition. In traditional logic, a proposition is divided into a subject and a predicate. For example, in the proposition "Socrates is a man," Socrates is the subject and man is the predicate. That is as far as the analysis goes. Russell is not satisfied with this because it leaves out one very important part of the proposition, namely, the verb. In order to eliminate this neglect of the verb, he chooses to analyse propositions into subjects and assertions. The assertion includes both the verb and the predicate. In "Socrates is a man," Socrates is the subject and "is a man" is the assertion.

Let us illustrate the difference between the two kinds of implication by an example which involves each. "If p implies q, then if p is true q is true" where p and q are symbols for propositions. In this statement, "p implies q" and "p's truth implies q's truth" both state material implications, while "p implies (material) q" formally implies "p's truth implies(Material) q's truth."

Formal implication holds between propositional functions, while material implication holds between propositions. Propositional functions are to be distinguished from propositions later, but let us point out the difference between the two at the present by an example of each. A proposition is of the form "Socrates is mortal." Notice that this statement contains no variable. Such a statement has to be true or false, and cannot be true at one time and false at another. The propositional function is of the form "X is mortal" for all values of X. X is a variable, and the statement is true or false only when a constant or fixed value is given to the variable.

2. The indefinable known as "the relation of a term (individual) to the class of which it is a member." This idea, as we have observed else where is denoted by the symbol $\in$. This relationship is distinguished from the relation of whole and part between classes. The distinction is the same as that between an individual and its species ($\in$) and the species and its genus. Let us illustrate. The relation of Socrates to the class of Greeks is the relation of an individual to the class of which it is a member; whereas the relation of the part to the whole between classes is that of the relation of Greeks to men. This distinction was first made by Frege and then by Peano. It was from Peano that Russell got the idea. Another distinction must be made, namely, the distinction between class and class-concept, or the predicate by which the class is defined. Hence men is a class, while man is a class-concept. The class must be considered collectively (for example, men, not man), if the relation $\in$ is to hold. The relation of part and whole between classes is transitive, but the relation between the individual and class of which it is a member is not.

3. The Notion of such that. Little can be said about this notion. About all that we can do is to point

out examples ot it. Russell says that it "is roughly equivalent to <u>who</u> or <u>which</u>, and represents the general notion of satisfying a propositional function."[1] Peano defined the notion of <u>such that</u> by the proposition: "the x's such that x is <u>a</u> are the class <u>a</u>." But Russell points out[2] that the class derived from <u>such that</u> is the genuine class in extension and as many, while <u>a</u> in "s is an <u>a</u>" is not the class, but the class-concept. Furthermore, Peano's definition involves the notion of <u>such that</u> itself and is therefore guilty of the vicious circle fallacy. An example of what is meant by the indefinable <u>such that</u> may be given. In the propositional function ϕx, the notion of <u>such that</u> is the relation between certain values of x which renders the propositional function either true or false. This is fundamentally the same thing as Peano said, but Russell does not call this a definition.

4. <u>The notion of relation.</u> The fourth indefinable is the notion of relation, and, yet, each of the preceding three have been explained, if not defined, as being relations. Because of this we would have to consider Russell's judgment concerning the indefinability of the above three and the order of these four as unsound, if it were not for the fact that he clearly states that his selection of fundamental indefinables is largely arbitrary. But we are now concerned with the notion of relation itself. We are told that "a relation between two terms is a con-

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1. <u>Principles</u>, p. 83.
2. <u>Ibid</u>, p. 82.

cept which occurs in a proposition in which there are two terms not occurring as concepts, and in which the interchange of the two terms gives a different proposition."[1] There are two parts of this explanation (we say "explanation" since it is not regarded as a definition) which needs further elucidation. The first is the meaning of the words "term" and "concept." "I call a <u>term</u>," Russell says, "whatever may be an object of thought, or may occur in any true or false proposition, or can be counted as <u>one</u>."[2] Hence it is synonymous with the words unit, individual, and entity (as Russell uses it). By making <u>term</u> synonymous with unit and individual he emphasizes the fact that every term is <u>one</u>, while with entity he means that every <u>term</u> has being, or <u>is</u> in some sense. Thus <u>term</u> is regarded as a very general word. Russell distinguishes between two kinds of terms, namely, things, or those which are indicated by proper names, and concepts, which are those indicated by all other words. Here proper names are to be understood to include all perticular points and instances, and many other entities not usually designated in this manner. Among concepts, two kinds are distinguished, namely, those indicated by adjectives, which are known as predicates or class-concepts, and those indicated by verbs, which are always or almost always relations.

Since we have cleared up the meaning of the words "term" and "concept," we proceed to the second diffi-

1. <u>Principles</u>, p. 95.
2. <u>Ibid</u>, p. 43.

culty in the above explanation of relation. This has to do with the last part, which says "the interchange of the two terms gives a different proposition." This clause is inserted to distinguish a relational proposition from one of the form "a and b are two," which is identical with "b and a are two." A relational proposition is symbolized in this manner: "aRb," where a and b are terms and R is the relation between them. The latter part of the definition asserts that to interchange the terms in the proposition "aRb" we would have "bRa," which is a new proposition. "It is characteristic of the relation of two terms that it proceeds, so to speak, <u>from</u> one to the other. This is what may be called the <u>sense</u> of the relation."[1] This characteristic is the source of order and series, which is very important in mathematical philosophy. The term from which the relation proceeds is known as the "referent" and the term to which it proceeds is known as the "relatum." While the relation between a and b is denoted by R, the relation between b and a, the latter being the converse of the former, is denoted by $\breve{R}$. One thing more about relations, namely, some relations hold between a term and itself, and such relations are not necessarily symmetrical.

5. <u>The Notion of a propositional function.</u> We have already illustrated the idea of a propositional function. At this point we will only say a few words in explanation of it. Russell explains, not defines, this

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1. <u>Principles</u>, p. 95.

notion in this manner: "ϕx is a propositional function if, for every value of x, ϕx is a proposition, determinate when x is given."[1] A function is "every relation which is many-one," that is "every relation for which a given referent has only one relatum."[2] This defines a function, "but when the function is a proposition, the notion involved is pre-supposed in the symbolism, and cannot be defined by means of it without a vicious circle: for in the above general definition of a function propositional functions already occur."[3]

6. <u>The notion of class.</u> To make clear what is meant by class and to distinguish it from all the other notions which are allied with it is regarded by Russell as one of the most difficult and important problems involved in mathematical philosophy. And it is this matter which in-volved him in contradictions for which he was not able to find an adequate solution in the <u>Principles.</u>

In the truest sense of the word, "class" is not an indefinable notion, but it is a fundamental notion of which Russell is not able to give a satisfactory definition. His difficulty is not the problem of the vicious circle in this instance, but it is a matter of making distinctions between allied ideas and in defining the term so that it will include all that it must include in order for it to fit his mathematical philosophy. For example, he can de-fine class adequately as far as finite classes are con-

1. Principles, p. 19.
2. <u>Ibid</u>, p. 83.
3. <u>Ibid</u>, p. 83.

cerned, but he wants a definition of class that will include infinite classes, finite classes, and the null-class.

There are two ways of defining class, namely, intentionally and extentionally. The intentional view defines class in relation to the predicates attached exclusively to a certain subject. From this view, classes are derived in this manner: Socrates is human; Socrates has humanity; Socrates is a man, and, lastly, Socrates is one among men. Only the last proposition explicitly contains the class as a constituent, but the others give rise to the class. The extentional view defines a class by the enumeration of its terms. The intentional view defines the kind of concept which denotes a class, whereas the extentional view defines the kind of object which is a class. The intentional view deals with the concept of a class, like <u>men</u>, which is not a class but it is a concept which denotes the class composed of men.

At this point it is necessary for us to observe the distinction which Russell makes between <u>class-concept</u> and the <u>concept of a class</u>. In common usage, these two expressions would mean the same, but not so with Russell. We can best point out this distinction by illustrations. The term <u>man</u> is regarded as a class-concept. This does not denote anything. When we merely say <u>man</u> we neither denote <u>one</u> man, <u>any</u> man, <u>some</u> man, <u>a</u> man, <u>the</u> man, nor a class known as man. It is only a class-concept. <u>Men</u> (or <u>all men</u>, the two being used synonymously by Russell), on the other hand, is the concept of a class, becuase it denotes. The object which it denotes is the class composed of men (all men).

The extentional view, as we have said, defines the kind of object which is a class, and this it does by enumeration of the terms, or members, of the class.

Although the general notion of class can be defined in this two-fold manner, that is by intention and by extention, from the point of view of practicality, infinite classes cannot be defined extentionally, because it is impossible to enumerate all of their terms, expressing a conjunction relationship between each term and another. Also the extentional definition eliminates the null-class. However, Russell believes that the definitions given by the two means are on a par logically. The difference, he says, is purely psychological. Nevertheless, Russell concludes that a class is essentially to be interpreted in extention, for it is either a term, or that kind of combination of terms which is indicated when terms are related by conjunction. But, practically, not logically, this extentional method is applicable only to finite classes. Whereas the intentional method is applicable to all classes, both finite and infinite. These are obtained as the objects denoted by concepts of classes (or the plurals of class-concepts), such as men, numbers, points, etc. It is concluded that the null-class, which has no terms, is a fiction, but it is maintained that there are null-class concepts.

The most important general conclusion in the discussion of classes is that "although any symbolic treatment must work largely with class-concepts and intention, classes and extention are logically more fundamental for the principles of mathematics."[1]

A grave contradiction arises out of this treatment of class. It has to do with regarding a class as one and as many. The human race is a class-concept as one, and men is the concept of a class as many. The contradiction is stated in this manner: "if w be the class of all classes which as single terms are not members of themselves as many, then w as one can be proved both to be and not to be a member of itself as many."[2] This has come to be known as "The Russell Paradox." Attempts at a solution of this and similar paradoxes gave rise to the theory of types, which is to be discussed later. This theory is hinted at in the body of the work on Principles, and a rough embryonic sketch of the theory is included in an appendix to this work in 1903. But the matter of its development is not our concern at the present.

As we observed at the beginning of the discussion of classes, the general notion of class is, in the last analysis, not regarded as an indefinable. He says that the notion of class in general can be "replaced as an indefinable, by that of the class of propositions defined by a propositional function."[3]

1. Principles, p. 81.
2. Ibid, p. 107.
3. Ibid, p. 106, f.n.

7. <u>The notion of denoting.</u> This notion has to do with class-concepts and their objects, or what they "denote." For example, it is false to say that "man is mortal." Man never dies, but men do. The distinction here is that between a concept and its object. The relationship between these is known as denotation. There are six denoting words used with class-concepts, namely, <u>all</u>, <u>every</u>, <u>any</u>, <u>a</u>, <u>some</u>, and <u>the</u>. "When a class-concept, preceded by one of the six words...occurs in a proposition, the proposition is, as a rule, not <u>about</u> the concept formed of the two words together, but about an object quite different from this, in general not a concept at all, but a term or complex of terms."[1] Russell decides that denoting is a perfectly definite relation, being the same in all six cases, namely, <u>a</u> thing, <u>any</u> thing, <u>some</u> thing, <u>all</u> things, <u>every</u> thing, and <u>the</u> thing. The difference in each of these cases is the nature of the denoted object and the denoting concept, but the denoting relationship is the same in all six instances.

8. <u>Any</u> or <u>every term</u>. <u>Any</u> seems to be half way between a conjunction and a disjunction. Russell calls it the <u>variable</u> conjunction. <u>Any term</u> denotes only one term, but it is completely irrelevant which it denotes, and what is said will be equally true whichever it may be. Using the symbol <u>a</u> for <u>term</u>, Russell says: "<u>any a</u> denotes a

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1. <u>Principles</u>, p. 64.

variable a, that is, whatever particular a we may fasten upon, it is certain that any a does not denote that one; and yet of that one any proposition is true which is true of any a." Any term is a concept denoting the true variable, and from a formal point of view the variable is the characteristic notion of mathematics. Every term is related to any term in that any proposition that is true of any a is also true of every a. Yet any a denotes an entirely different object than every a does. Every a denotes all the a's, but it does it severally rather than collectively. Any a does not denote any particular a, but what is said of any a is true of every a.[1]

Later in the Principles Russell introduces term and a term as indefinables. These are closely related to what is included in any term as explained above.

As we have seen, in the Principles Russell regarded the primitive ideas (indefinable logical constants) as being eight or nine or perhaps ten or maybe eleven, depending on whether class is regarded as indefinable in the last analysis, and whether term, a term, any term, and every term are to be regarded as each being an indefinable. If we eliminate class and every term, the former being definable in a sense and the latter being mentioned as an indefinable in a summary statement[2] but not used, we have nine indefinables. If we maintain class and consolidate

1. Principles, p. 152.
2. Ibid, p. 106.

term, a term, any term, and every term into one, then we have eight indefinables. What seems to be correct is to eliminate class and term, since these are defined in the book, in spite of the fact that they are contained in some of his lists of indefinables, and to maintain a term, any term, and every term. Then we have nine indefinables. It is possible to eliminate every term since he does not make important use of it. Then there would be eight indefinables. Therefore, we conclude that Russell is right in the first of the book[1] when he says that the number of indefinable logical constants appear to be eight or nine. When we say that he is right, we mean that he is stating the number of indefinables which he actually proposed within the pages of the book. As we shall see, he greatly reduces this number in subsequent work.

These are all the indefinables regarded as necessary for the deduction of all of mathematics from them along with a few indemonstrable propositions, according to the Principles.

The Indemonstrable Propositions of the Principles

Concerning the indemonstrable propositions, Russell says: "in regard to some of them I know of no grounds for regarding them as indemonstrable except that they have

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1. Principles, p. 11.

hitherto remained undemonstrated."[1] These indemonstrable propositions are regarded as twenty in number, ten axioms of deduction and ten other axioms of a general logical nature. The first ten listed will be the axioms of deduction, which means that they come under the calculus of propositions.

(1). "If p implies q, then p implies q," or "Whatever p and q may be, 'p implies q' is a proposition." (2). "If p implies q, then p implies p," or "whatever implies anything is a proposition." (3). "If p implies q, then q implies q," or "whatever is implied by anything is a proposition." (4). "A true hypothesis in an implication may be dropped, and the consequent asserted." This, Russell believed in 1903, to be incapable of symbolic statement. (5). The principle of <u>simplification</u>, which is "if p implies p and q implies q, then pq implies p," or "the joint assertion of two propositions implies the assertion of the first of the two." (6). The principle of the <u>syllogism</u>, which may be stated "if p implies q and q implies r, then p implies r." (7). The principle of <u>importation</u>, which is stated this way: "If q implies r implies r, then pq implies r," or "if p implies that q implies r, then r follows from the joint assertion of p and q." (8). The principle of <u>exportation</u>, which is the converse of no. 7. It is stated: "If p implies p and q im-

1. <u>Principles</u>, pp. 15-16.

plies q, then if pq implies r, then p implies that q implies r." (9). The principle of composition: "If p implies q and p implies r, then p implies qr," or "a proposition which implies each of two propositions implies them both." (10). The principle of reduction: "If p implies p and q implies q, then '"p implies q" implies p.'"

The other ten primitive (indemonstrable) propositions are found in the calculus of classes and the calculus of relations, the first two,[1] as listed, in the former and the last eight[2] in the latter. (11). "If x belongs to the class of terms satisfying a propositional function ϕx, then ϕx is true." (12). "If ϕx and ψx are equivalent propositions for all values of x, then the class of x's such that ϕx is true is identical with the class of x's such that ψ x is true." (13). "xRy is a proposition for all values of x and y" where xRy is an expression of the propositional function "x has the relation R to y." (14). Every relation has a converse, i.e. that, "if R be any relation, there is a relation R' such that xRy is equivalent ot yR'x for all values of x and y." (15). "Between any two terms there is a relation not holding between any two other terms." (16). The negation of a relation is a relation. (17). The logical product of a class of relations is a relation. (18). The relative pro-

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1. Principles, p. 20.
2. Ibid, pp. 24-26.

duct of two relations is a relation. (19). Material implication is a relation, and (20). the relation of a term to the class of which it is a member is a relation.

With these fundamental principles (the indefinables and the indemonstrable propositions, or primitive ideas and primitive propositions, as they are called in Principia Mathematica) Russell proceeds to work out the fundamental concepts of mathematics. He defines disjunction, or logical addition, negation, identity, number, infinity, continuity, the various spaces of geometry, motion, etc., and asserts that the laws of contradiction, excluded middle, double negation, and all of the formal properties of logical multiplication and addition--the associative, commutative and distributive laws--can be demonstrated. These concepts will be considered in due time, but let us now consider how these primitive ideas and propositions were modified as he further developed his thought on the subject.

Primitive Ideas in 1908

In 1908 Russell published an article entitled "Mathematical Logic as based on the Theory of Types."[1] At this time he had reduced his list of primitive ideas to seven,[2] but in this list there were some which had been

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1. American Journal of Mathematics, Vol. 30.
2. Ibid, pp. 244-245.

defined in 1903 and many of the ones of 1903 were omitted. These we will indicate as we enumerate them. (1). The propositional function. He states it in this manner: "Any propositional function of a variable x or of several variables x, y, z." This he symbolized as ϕx or $\phi(x, y, z\ldots)$. (2). The negation of a proposition, which is denoted by $\sim p$, where p is a proposition. In the Principles,[1] Russell defined negation as "not-p is equivalent to the assertion that p implies all propositions, i.e. that 'r implies r' implies 'p implies r' whatever r may be." (3). Disjunction, or the logical sum of two propositions, denoted by $p \vee q$. In the Principles,[2] he was able to give two definitions of this idea. The first was: "'p or q' is equivalent to '"p implies q" implies q'", and the second: "Any proposition implied by p and implied by q is true," or, in other words, "'p implies s' and 'q implies s' together imply s, whatever s may be." (4). The truth of any value of a propositional function, namely, the truth of ϕx where x is not specified. (5). The truth of all values of a propositional function, which is denoted: $(x).\phi x$, or $(x):\phi x$, or whatever larger number of dots may be necessary to bracket off the proposition. In $(x).\phi x$, x is called an apparent variable, whereas, when ϕx is asserted, when x is not specified, x is a real variable. These two indefinables,

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1. p. 18.
2. p. 17.

numbers 4 and 5, have to do with the quality of truth, which is mentioned in one list of the indefinables in the Principles,[1] where it is spoken of as a fundamental notion in mathematics but is said not to be a constituent of the mathematical propositions, which is quite true. The principle is neglected in the former work. (6). Any predicative function of an argument of any type. This is represented by $\phi!x$, or $\phi!\alpha$, or $\phi!R$, according to the circumstances. Russell says that "a predicative function of x is one whose values are propositions of the type next above that of x, if x is an individual or a proposition, or that of values of x, if x is a function. It may be described as one in which the apparent variables, if any, are all of the same type as x or of lower type; and a variable is of lower type than x if it can significantly occur as argument to x, or as argument to an argument to x, etc." This explanation presupposes knowledge of the theory of types, which has to be explained later. (7). Assertion, which is represented by $\vdash$. It is the assertion that some proposition, or that any value of some propositional function, is true.

In this list we notice that implication, the relation of a term to the class of which it is a member, relation, the notion of such that, denoting, a term, any term, and every term, all of which are regarded as indefinables in the Principles are omitted. The two lists con-

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1. p. 3.

tain only one in common and that is the notion of a propositional function.

Primitive Propositions in 1908

In the same article mentioned above, the number of primitive propositions, which was twenty in the *Principles*, has been reduced to fourteen, and these are expressed largely in the symbolism which was later to characterize *Principia Mathematica*. We shall list these propositions as they are given.[1]

But first, we must explain the symbols which are used. (1). $\vdash$ means assertion, or "it is asserted that." (2). $\vee$ means disjunction, or *or*. (3). $\cdot$ means conjunction, or *and*. (4). $\supset$ means implication, or *if, then* relation. (5). ":" used for brackets, or when necessary "." or ":." are used to distinguish the section to be bracked. (6). ϕx means that the variable x has the ~~function of~~ quality ϕ. The expression is a propositional function when x is any x. (7). $(x).\phi x$ means that x has the quality ~~function~~ ϕ for all values of x. For example: x is mortal is written ϕx, where ϕ means mortal, and then $(x).\phi x$ means that for all values of x, x is mortal. (8). $\phi\hat{x}$ means that x in general has the quality ~~function~~ ϕ. Instead of *all* x's as in $(x).\phi x$, it means *any* x. (9). We might point out that Roman letters are used for the variables and Greek letters for the qualities ~~functions~~. (10). "f" means function like

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1. *American Journal of Mathematics*, vol. 30, pp. 246-248.

$f(\phi x)$. (11). $(\exists x).\phi x$ means ϕx is true for some values of x, which asserts that there exists one x with the ~~function~~ quality ϕ. This is what is called an existence theorem. Also $(\exists f)$ means some function, or there is one function, so on. (12). $f!x$ means that this is an elementary function, which means that its argument x is an individual. This constitutes a function of the first order of of the first type. (13). $\equiv$ means equivalence.

We now return to the list of primitive propositions given in the 1908 article. They are as follows. (1). "A proposition implied by a true premise is true." This axiom is not given symbolic expression. It has to do with the indefinable quality of truth. (2). $\vdash p \vee p . \supset . p$., which means that if it is asserted that p or p is true, then p is true. This is called the "principle of tautology" in <u>Principia Mathematica</u>. (3). $\vdash : q . \supset p \vee q$., which asserts that if q is true, then p or q is true, or, in other words, if a proposition is true, then an alternative proposition can be joined on to it and at least one of the two will be true. For example: if it is hot, we can add that it is hot or cold, and it will be true that it is hot. (4). $\vdash : p \vee q . \supset . q \vee p$., which states that "If p or q is true, q or p is true." It is known as the principle of "permutation" in <u>Principia.</u> It merely states that a disjunction is symmetrical--that it may be taken in either of its two possible orders without affecting its truth. The same might be said of a conjunction, but it is not necessary to regard this as a primitive proposition. (5). $\vdash$: $p \vee (q \vee r) . \supset .$ $q \vee (p \vee r)$., which states that "If either p is true or q or r is true, then either q is true, or p or r is true." This is known as the "associative principle" for disjunction. (6). $\vdash$:. $q \supset r . \supset : p \vee q . \supset . p \vee r$. (The two dots after the first horseshoe sign serves to bracket off the whole expression following.), which states that "If q implies r, then p or q implies p or r." This is known as the principle of "summation" since it states that the addition of the same alternative to both the condition and the consequent of an implication does not affect its truth. (7). $\vdash$: $(x) . \phi x . \supset . \phi y$, which asserts that "if all values of $\phi \hat{x}$ are true, then ϕy is true, where ϕy is any value." (8). "If ϕy is true, where ϕy is any value of $\phi \hat{x}$, then $(x) . \phi x$ is true." This, Russell says, is incapable of strict symbolic expression. (9). $\vdash$: $(x) . \phi x . \supset \phi a$, where a is any definite constant. This asserts that if it is true that x has the quality ϕ for all values of x, then any definite constant has the quality ϕ. (10). $\vdash$:. $(x) . p \vee \phi x . \supset :$ $p . \vee . (x) . \phi x$, which, being interpreted, says that "if 'p or ϕx' is always true, then either p is true, or ϕx is always true.[1] (11). "Where $f(\phi x)$ is true whatever argument x may be, and $F(\phi y)$ is true whatever possible argument y may be, then $\{f(\phi x) . F(\phi x)\}$ is true whatever possible argument x may be." This is known as the axiom of

1. As used here, "is always true" means that it is true in every case.

the "identification of variables." (12). If $\phi x \supset \psi x$ is true for any possible x, then ψx is true for any possible x." (13). $\vdash :. (\exists f) :. (x) : \phi x . \equiv . f!x$. This is the axiom of reducibility, which is introduced in the theory of types. It states that, given any function ϕx, there is a predicative function $f!x$ such that $f!x$ is always equivalent to ϕx. (14). $\vdash :. (\exists f) :. (x,y) : \phi(x,y) . \equiv . f!(x,y)$. This is the axiom of reducibility for functions with two or more variables.

These primitive propositions are said to be equally applicable to all types without involving contradictions. What is meant by types will be explained under a section devoted to the theory of types.

Primitive Ideas of Principia Mathematica

The first volume of Principia was published in 1910. Russell and Whitehead had been working together on it since 1900. In this work the number of indefinables or primitive ideas is ten. They are: (1). The idea of an "elementary proposition," by which is meant a proposition which does not involve any variables, or, in other words, one which does not involve words like "all," "some," "the," or their equivalents. (2). Elementary propositional functions. "By an 'elementary propositional function,'" the authors say, "we shall mean an expression containing an undetermined constituent, i.e. a variable, or several such constituents, and such that, when the undetermined constituent or constituents are determined, i.e. when values are assigned to the variable or variables, the resulting value of the expression in question is an elementary proposition."[1] (3). Assertion. A proposition may be asserted or it may be merely considered. For examples: "Caesar died" is asserted. But in the statement, "Caesar died is a proposition," the part "Caesar died" is no longer an assertion. It is merely considered, but the whole proposition is an assertion. (4). Assertion of a propositional function. Where ϕx is a propositional function, we can assert it without assigning a value to x. (5). The idea of negation. If p is a proposition, the negation of it is the proposition "not-p," which is symbolized in this manner: $\sim p$. (6). Disjunction. This is what is known as the logical sum of two propositions, which is expressed "p or q," which means that either p is true or q is true, or p is false, or q is false, or anything else that you want to assert about them.[2] These six indefinables occur in the theory of deduction, which constitutes the calculus of propositions. The following three[3] occur in the theory of apparent variables. (7). The idea expressed by ϕx is always, meaning in all cases, true, which is symbolized in this manner: $(x).\phi x$.

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1. Principia, Vol. 1, p. 96. All references to Principia, unless it is indicated otherwise, will refer to the first edition.
2. Ibid, Vol. 1, pp. 95-96.
3. Ibid, Vol. 1., p 32 for the first two propositions, and pp. 138, 53-54 for the third.

(8). The idea expressed by ϕx is sometimes (in some cases) true, which is denoted by the notation: $(\exists x). \phi x$. Here "$\exists$" stands for "there exists," and the whole symbol may be read "there exists an x such that x has the quality ϕ." (9). The idea of an individual. An x is "individual" if it is neither a proposition nor a function. Individuals are genuine constituents of propositions and do not disappear on analysis as classes and descriptions do. (10). Matrix, or predicative function.[1] This idea occurs in the theory of types. The name "matrix" is given to any function regardless of the number of variables, which does not involve any apparent variables. A function is said to be predicative when it is a matrix.

This list of indefinables omits one of the primitive ideas included in the 1908 list and four new ones are added. The one of the former list that is omitted is number 4 of the list as given above, which was: "The truth of <u>any</u> value of a propositional function," namely, of ϕx where x is not specified. The four new ones are the following numbers in the above list: 1, 4, 8, and 9.

<u>The Primitive Propositions of Principia</u>

In the 1908 article, Russell needed only fourteen primitive propositions, but the *Principia* requires eighteen. The eighteen are the following.[2] (1). *1.1.

1. *Principia*, p. 172.
2. The numbers with an asterisk before them are for reference in the *Principia*.

"Anything implied by a true elementary proposition is true." (2). *1.11. "When ϕx can be asserted, where x is a real variable, and $\phi x \supset \psi x$ can be asserted, where x is a real variable, then ψx can be asserted, where x is a real variable." This is true also for functions of several variables. (3). *1.2. $\vdash : p \vee p . \supset . p$, which states: "If either p is true or p is true, then p is true." (4). *1.3. $\vdash : q . \supset . p \vee q$. (5). *1.4. $\vdash : p \vee q . \supset . q \vee p$. (6). *1.5. $\vdash : p \vee (q \vee r) . \supset . q \vee (p \vee r)$. (7). *1.6. $\vdash :. q \supset r . \supset : p \vee q . \supset . p \vee r$. (8). *1.7. "If p is an elementary proposition, $\sim p$ is an elementary proposition." (9). *1.71. "If p and q are elementary propositions, $p \vee q$ is an elementary proposition." (10). *1.72. "If ϕp and ψp are elementary propositional functions which take elementary propositions as arguments, $\phi p \vee \psi p$ is an elementary propositional function." (11). *9.1. $\vdash : \phi x . \supset . (\exists z) . \phi z$, which states that if ϕx is true, then ϕz is true in some cases, that is at least in one case. This primitive proposition gives the only method of proving "existence-theorems." (12). *9.11. $\vdash : \phi x \vee \phi y . \supset . (\exists z) . \phi z$, which states that if ϕx or ϕy is true, then ϕz is true in some cases. (13). *9.12. "What is implied by a true premise is true." This is the extention of *1.1 to propositions which are not elementary. (14). *9.13. "In any assertion containing a real variable, this real variable may be turned into an apparent variable of which all possible values are asserted to satisfy the function in question." The following four propositions have to do with the theory of types. (15). *9.14. "If 'ϕx' is significant, then if x is of the same type as a, 'ϕa' is significant, and vice versa." In this proposition there are two terms that need clarification. The first is "is significant." It means that some functions have meaning and some do not. One that is significant is within a type that gives it meaning. The second term needing clarification is "being of the same type." The following is a definition of this term. "We say that u and v 'are of the same type' if (a) both are individuals, (b) both are elementary functions taking arguments of the same type, (c) u is a function and v is its negation, (d) u is $\phi\hat{x}$ or $\psi\hat{x}$, and v is $\phi\hat{x} \vee \psi\hat{x}$, where $\phi\hat{x}$ and $\psi\hat{x}$ are elementary functions, (e) u is $(y) . \phi(\hat{x}, y)$ and v is $(z) . (\hat{x}, z)$, where $\phi(\hat{x}, \hat{y})$, $(\hat{x}, \hat{y})$ are of the same type, (f) both are elementary propositions, (g) u is a proposition and v is $\sim u$, or (h) u is $(x) . \phi x$ and v is $(y) . \psi y$, where $\phi\hat{x}$ and $\psi\hat{x}$ are of the same type."[1] (16). *9.15. "If, for some a, there is a proposition ϕa, then there is a function $\phi\hat{x}$, and vice versa." The following two primitive propositions are the two forms of the axiom of reducibility, which will receive further treatment later. The two propositions are: (17). *12.1. $\vdash : (\exists f) : \phi x . \equiv_x . f!x$. (18). *12.11. $\vdash : (\exists f) : \phi(x,y) . \equiv_{x,y} . f!(x,y)$. These two axioms of reducibility state that "any function of one or two variables is formally equivalent

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1. *Principia*, p. 138.

to some predicative function of one or two variables, as the case may be. Of the two axioms, the first is chiefly needed in the theory of classes, and the second in the theory of relations. They are slightly modified from the form in which they appeared in 1908.

Of these eighteen primitive propositions in the Principia, only six are taken without change from the list of fourteen in the 1908 article. These six are the following:[1] (3). *1.2. is number 2 in the 1908 list; (4). *1.3, number 3 in 1908 list; (5). *1.4, number 4 in the former list; (6). *1.5, number 5 in former list; (7). *1.6, number 6 in 1908 list; (13). *9.12, number 1 in 1908 list. Numbers 13 and 14 of the 1908 list, which are the two axioms of reducibility, are reproduced with slight modifications in numbers (17). *12.1 and (18). *12.11. We state them here side by side so that the difference may more easily be observed. The first of the two was in this form in 1908: $\vdash :. (\exists f) :. (x) : \phi x . \equiv f!x$, and like this in Principia: $\vdash : (\exists f) : \phi x . \equiv_x . f!x$. The second proposition appeared in 1908 as: $\vdash :(\exists f) :. (x,y) : \phi(x,y) . \equiv f!(x,y)$; and it was modified to this in Principia: $\vdash : (\exists f) : \phi(x,y) . \equiv_{x,y} . f!(x,y)$. These modifications do not constitute a change in meaning, but only in notation.

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1. We are giving both the numbers as we listed above and the numbers in Principia.

Later Modifications of the Primitive Ideas

In *Principia*,[1] we have stated what the authors regard as the four fundamental functions of propositions. They are negation, disjunction, conjunction, and implication. The first two of these are regarded as primitive ideas and the other two are defined in terms of them. Dr. H. M. Sheffer[2] has added a fifth function to this list, which is the idea of the incompatibility of two propositions. It states that both p and q cannot be true. The idea of incompatibility is denoted by what is called the stroke, which is symbolized in this way: "/." Thus we get p/q. Sheffer regards this idea as primitive, and he defines the other four functions in terms of it. Negation is defined as the incompatibility of a proposition with itself, which is denoted "p/p." Disjunction is the incompatibility of ~p and ~q, which is (p/p)|(q/q). Implication is the incompatibility of p and ~q, which is p|(q/q). And conjunction is the negation of incompatibility, which is (p/q)|(p/q).

In the *Introduction to Mathematical Philosophy*[3], Russell adopts this method of Sheffer, and thus substitutes the primitive idea of incompatibility for the two primitive ideas of negation and disjunction. This method is adopted also in the second edition of *Principia*.[4]

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1. Vol. 1., p. 6.
2. See the *Transactions of the American Mathematical Society*, Vol. xiv, pp. 481-488.
3. p. 148.
4. *Principia* (2nd. edition), Vol. 1, p. xiii.

The second edition (1925) of the Principia does away also with the primitive idea of "assertion of a propositional function, which is number 4 in our list given above. Thus three primitive ideas that appeared in the first edition are eliminated in the second. They are the ideas of negation, disjunction, and the assertion of a propositional function. But one new one is added, namely, the idea of the incompatibility of two propositions. This gives eight primitive ideas in the second edition of Principia.

In the introduction to the second edition of the Principles (1938),[1] Russell makes the following statement: "After the utmost efforts to reduce the number of undefined elements in the logical calculus, we shall find ourselves left with two (at least) which seem indispensable: one is incompatibility; the other is the truth of all values of a propositional function." We do not have the work in which he was able to work out the whole of the logical calculus with only these two primitive ideas, and this is highly regretable. But it would appear from this statement that he has been able to reduce the primitive ideas necessary for his logical calculus and all of mathematics to two.

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1. p. xi.

Later Modifications of the Primitive Propositions

M. Jean Nicod[1] replaced five of the primitive propositions in the first edition of Principia with only one. The ones replaced were the following;[2] (3). *1.2.; (4). *1.3.; (5). *1.4.; (6). *1.5.; (7). *1.71. These five are the formal indefinables in the theory of deduction (the propositional calculus). Let us observe how Nicod was able to reduce these to the one sole formal principle of deduction. Taking advantage of Sheffer's substitution of the idea of incompatibility as the primitive idea by which negation, implication, conjunction, and disjunction can be defined, he deduced his one formal principle of deduction in the following manner. We have already seen that $p|(q/q)$ means "p implies q." And $p|(q/r)$ means "p implies both q and r," for this expression means "p is incompatible with the incompatibility of q and r. Next observe that $t|(t/t)$ means "t implies itself," which is a particular case of $p|(q/p)$. Writing the negation of p as $\overline{p}$ and of p/s as $\overline{p/s}$, which is the conjunction of p and s, we have $(s/q)|\overline{p/s}$, which expresses the incompatibility of s/q with the conjunction of p and s. In other words, it says that if p and s are both true, s/q is false,

1. See "A Reduction in the Number of the Primitive Propositions of Logic," Proceedings of the Cambridge Philosophical Society, Vol. xix.
2. We are giving the number in our list above and the number in Principia.

i.e. s and q are both true. Or again, it states that p and s together imply s and q together. Now, put p for $p\,(q/r)$; π for $t|(t/t)$; Q for $(s/q)|\overline{p/s}$. Then we have Nicod's only formal principle of deduction, which is $P|\pi/Q$, which states that P implies both π and Q.

In addition to this one formal principle, Nicod employs one non-formal principle which belongs to the theory of types and one similar to the principle that, given p, and given that p implies q, we can assert q. This principle is stated in this manner: "If $p|(r/q)$ is true, and p is true, then q is true."

From these three primitive propositions, one formal and two non-formal, Nicod deduces the whole theory of deduction, with the exception of the deduction from or to the universal truth of propositional functions.

In the Introduction to Mathematical Philosophy,[1] Russell seems to accept this reduction of primitive propositions in the theory of deduction to one formal and two non-formal ones. However, in the second edition of Principia, only one of the original five non-formal primitive propositions is done away with, which is (2). *1.11. Nicod's one formal proposition is substituted for the five formal ones of the first edition.

The authors, in the second edition of Principia, are very much dissatisfied with the two axioms of reducibility, which are numbers (17) and (18) in our list above

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1. Pp. 151-152.

and numbers *12.1 and *12.11 in Principia. Concerning it, they say that it "has a purely pragmatic justification: it leads to the desired results, and to no others. But clearly it is not the sort of axiom with which we can rest content. On this subject, however, it cannot be said that a satisfactory solution is as yet obtainable."[1]

Wittgenstein, a former student of Russell, suggested a solution in his book Tractatus Logico-Philosophicus (*5.54ff), which assumes that functions of propositions are always truth-functions, and that a function can only occur in a proposition through its values.

Russell and Whitehead have worked out this theory,[2] but, although they have considered it worth working out its consequences, they do not regard it as certainly right. There are difficulties with the theory, but Russell and Whitehead say that "perhaps they are not insurmountable." One chief objection is that it requires all functions of functions to be extentional. This the authors are not prepared to accept. With the adoption of this theory practically everything of Vol. 1 of Principia remains true, although new proofs are often required; the theory of inductive cardinals and ordinals remain; but the theory of infinite Dedekindian and well-ordered series largely give way, and, consequently, irrationals and real numbers gen-

1. Principia (2nd. edition), Vol. 1, p. xiv.
2. Ibid (2nd. edition), Vol. 1, Appendix C.

erally cannot be dealt with adequately. Also Cantor's proof that $2^n > n$ collapses unless n is finite. The authors say that "perhaps some further axiom, less objectionable than the axiom of reducibility, might give these results, but we have not succeeded in finding such an axiom."[1] But tentatively the two primitive propositions, namely, (1) "functions of functions are always truth-functions" and (2) "a function can only occur in a proposition through its values"--, are substituted in this appendix[2] for the two axioms of reducibility. But there is no finality claimed for these. They are used only because they are the best that the authors can do.

Thus, in the second edition of Principia, the eighteen primitive propositions of the first edition have been reduced to thirteen with eight of the original ones abolished and three new ones added.

In the second edition of Principia,[3] we have all of mathematics, except geometry, deduced from eight undefined ideas and thirteen undemonstrated propositions. As we saw above, in 1938, Russell thought that the number of undefined ideas could be further reduced to only two. Concerning the deduction of geometry from these same primitive ideas and propositions, in the second edition of Principia, the authors promised a fourth volume in which the deduction of geometry would be demonstrated, but this volume has never

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1. Principia (2nd. edition), Vol. I, p. xiv.
2. Ibid, (2nd. edition), Vol. I, Appendix C.
3. The 2nd. edition of Principia differs from the first only in that the former has an added introduction and several appendices.

been published.

This brings us to the conclusion of our discussion of the logical calculus as such. These primitive ideas and primitive propositions constitute the calculus. All else is developed from them and by means of them either directly or indirectly. The only tests of the validity of any set of primitive ideas and propositions are: (1). the inability of further reduction of the number by defining or demonstrating one (or several) in terms of the others, and (2). the ability to deduce all additional necessary facts in logic and mathematics from them. It is evident from the way Russell has altered his primitive ideas and propositions from time to time that no finality can be claimed for any particular set that he has produced. It remains to be determined whether or not all of the necessary facts can be deduced by purely logical means from any one set of these ideas and propositions. We shall consider this in due time.

There remains one topic which might be discussed under the logical calculus, namely, the theory of types, which is set forth in the undemonstrated propositions of <u>Principia</u>, but we shall deal with it in the next chapter independently of the logical calculus because of its distinctive interest to broader philosophical questions.

Chapter V

The Logical Paradoxes and the Theory of Types

In ancient and medieval times much attention was given by philosophers to what is known as logical paradoxes, but in modern times these paradoxes fell into disrepute until they were rediscovered by the development of symbolic logic. As late as Lotze the problems involved were dismissed as insoluble without effort being exerted to change this state of affairs. However, in 1869, Mr. C. S. Peirce[1] considered the paradox resulting from the proposition, "This proposition is false." His treatment of the problem was very similar to that offered by Paulus Venetus in the Middle Ages, and he really did nothing to bring about the solution of the problem except in that his discussion put this and similar problems before philosophers again.

Little serious work was done in modern times on these paradoxes until Russell, in his treatment of classes in the Principles (1903), discovered what has come to be known as "Russell's Paradox." This pradox or contradiction is stated in the Principles[2] as follows: "If w be the class of all classes which as single terms are not members of

1. "The Validity of the Laws of Logic," Journal of Speculative Philosophy, Vol. II: no. 4.
2. P. 107.

themselves as many, then w as one can be proved both to be and not to be a member of itself as many."

This paradox is only one of a whole series of similar ones. In the Principia,[1] seven of the most significant of these contradictions are given as follows: (1). The first is the classic one about the liar. This is of the form: "Epimenides, the Cretan, said that all Cretans are liars." The simplest form of this contradiction is the statement: "I am lying." If this statement is true, then it must be false, and if it is false, then it is true. (2). Russell's paradox, which is stated: "Let w be the class of all those classes which are not members of themselves. Then, whatever class x may be, 'x is a w' is equivalent to 'x is not an x.' Hence, giving x the value of w, 'w is a w' is equivalent to 'w is not a w.'" (3). "Let T be the relation which subsists between two relations R and S whenever R does not have the relation R to S. Then, whatever relations R and S may be, 'R has the relation T to S' is equivalent to 'R does not have the relation R to S.' Hence, giving the value T to both R and S, 'T has the relation T to T' is equivalent to 'T does not have the relation T to T.'" (4). The next contradiction is one known as Burali-Forti's. It asserts that any series of ordinal numbers beginning with 0 has an ordinal number greater by one than the highest term of the series. Therefore, the

1. Vol. 1, pp. 63-64.

series of all ordinals would have an ordinal number greater than the highest ordinal, which would mean that the highest ordinal is greater than itself. (5). "The least integer not namable in fewer than nineteen syllables" (which Russell says is 111,777) is itself a name consisting of only eighteen syllables. (6). "Among transfinite ordinals some can be defined, while others cannot; for the total number of possible definitions is $\aleph_0$ [which is the number of finite integers] , while the number of transfinite ordinals exceeds $\aleph_0$. Hence there must be indefinable ordinals, and among these there must be a least. But this is defined as 'the least indefinable ordinal,' which is a contradiction. (7). The last is known as Richard's paradox. It is as follows:

> Consider all decimals that can be defined by means of a finite number of words; let E be the class of such decimals. Then E has $\aleph_0$ terms; hence its members can be ordered as the 1st, 2nd, 3rd...Let N be a number defined as follows. If the nth figure in the nth decimal is p, let the nth figure in N be p+1 (or 0, if p=9). Then N is different from all the members of E, since, whatever finite value n may have, the nth figure in N is different from the nth figure in the nth of the decimals composing E, and therefore N is different from the nth decimal. Nevertheless we have defined N in a finite number of words, and therefore N ought to be a member of E. Thus N both is and is not a member of E.

All of these contradictions are examples of what is known as the vicious-circle fallacy, that is the fallacy of self-reference or reflexiveness. Each of these propositions resulting in contradictions possess the characteristic of referring to itself. Each proposition is perfectly true or false without involving any form of contradiction except when it is applied to itself. In each proposition something is said about all cases of some kind, and from what is said a new case is generated, which both is and is not the same kind of case to which the all of the proposition refers. This new case, when it is included with the other cases, constitutes what Russell and Whitehead call "illegitimate totalities." In order to avoid the vidious-circle contradictions, it seems as if some method must be devised whereby the new case generated by a statement about the "all of a kind" can be prevented from joining with the other cases and thus forming an illegitimate totality. It is this that the theory of types seeks to do.

Most of the ideas in Principia seem to have originated with Russell. We draw this conclusion concerning the fundamental ideas of Principia because these ideas, for the most part, had been set forth by Russell previous to the publication of the Principia. In these previous works, Russell is very careful to acknowledge the sources of his ideas, and Whitehead, although Russell had worked in close contact with him since as early as 1900, does not feature largely in his acknowledgments. The importance of the contributions of Whitehead to Principia cannot be denied or belittled, but his greatest contribution seems to have been in the development of the symbolism and the symbolic demonstrations.

This conclusion certainly applies to the theory of types. Russell ran into the vicious-circle contradictions in the Principles, and the theory of types, in a rough and embryonic form, first appeared in an appendix to this work as a possible solution of the "Russell Paradox." In an article[1] in 1908, Russell presented the theory of types with all of the essential details which the theory has in the Principia. Our discussion of the theory is based on the form in which it appears in Principia,[2] since it is this form upon which the whole structure of Principia is based.

These paradoxes concern various kinds of objects: propositions, classes, cardinal and ordinal numbers, etc. From this we might think that we would need a theory of types for each of these kinds of objects, but this is not the case. Russell and Whitehead reduce statements that verbally concern classes and relations to statements concerning propositional functions by the theory of descriptions.[3] This theory employs what is know as "incomplete symbols," which have meaning only in a context, to describe, not define, a "so and so." Such descriptions (or incomplete symbols)are distinguished from proper names, in a generalized sense, in that they do not stand for certain

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1. "Mathematical Logic as based on the Theory of Types," American Journal of Mathematics, Vol. 30, pp. 222-262.
2. Principia, Vol. 1, pp. 39-68.
3. See chapter III of Introduction to Principia, Vol. 1, for a discussion of the theory of descriptions.

objects and thus have no meaning when in isolation. The following is an example of how a proposition about a relation can be broken down to a propositional function by the theory of descriptions. Take the proposition "Scott is the author of Waverley." There is no constituent corresponding to "the author of Waverley." This is an incomplete symbol or description. The proposition is broken down to the propositional function "'x wrote Waverley is equivalent to x is Scott' is true for all values of x." All propositions which verbally concern classes and relations can be broken down to propositional functions in this manner. Therefore, a theory of types based on propositional functions will also be applicable to collections of propositions, of classes, of cardinal and ordinal numbers, etc.

A propositional function, as we have already noted, is a statement which contains a variable x, and expresses a proposition when a definite value is given to x. Thus it differs from a proposition only in that a proposition is definite and a propositional function is indefinite or ambiguous. The function itself is that which ambiguously denotes, and an undetermined value of the function is that which is ambiguously denoted. The undetermined value is written "$\phi\hat{x}$," while the function itself is written "ϕx." Propositional functions are of two kinds, namely, those in which the truth-value[1] of the function is dependent

1. The truth-value is whether it is true or whether it is false, and thus falsity is as much a truth-value as truth is.

upon the value given to the variable and those in which the truth-value is the same for all values of the variable. The first type is of the form "x is a man;" the second type is of the form "'x is a man' implies 'x is mortal.'" Only functions of the latter type can be regarded as true or false and treated as propositions in logic. In this type the variable is called an apparent variable.

In his examination of the paradoxes, Russell concludes that all of them contain an apparent variable that refers to "all" of a totality which is illegitimate. The apparent variable is deceitful. The statement "for all values" is never completely justified, for sometimes there is one value that will result in a paradox. This value is the function itself. For example, take the proposition: "All propositions are false." This may be stated: "x is a proposition, and for all values of x, x is false." Now insert the proposition just stated as a value of x, and the result is a paradox. If the proposition is true, it is false. Russell condludes that such variables must have a range of values which result in significant propositions and that the function itself is excluded from this range in each case.

This exclusion of a function itself from the range of significance of a variable which occurs in it is accomplished by the theory of types. Russell sums up the technical essence of this theory as follows: "Given a propositional function 'ϕx' of which all values are true, there are expressions which it is not legitimate to substitute for 'x'," [1] which means that a logical type is the range of significance of a propositional function, and that the range of significance excludes the function itself. This exclusion is accomplished by the establishment of a series of ranges of significance. Thus a function cannot include itself in its range, but it is included in the next highest range in the hierarchy of functions. The first range is defined as those functions which assume the totality of "individuals," that is those objects which are not analyzable into term and predicate as are propositions. These constitute the first logical type and the functions in which they occur are known as first order functions. And functions which assume the totality of propositions or functions of the first order constitute what is known as the functions of the second order. In like manner, functions which assume the totality of functions or propositions of the second order constitute the third order, etc.

Now let us consider how this theory of types solves the paradoxes. As we have seen, all of these paradoxes possess one common characteristic which is responsible for the paradox, and this characteristic is self-reference. The paradox results when these propositions with apparent variables about "all of a kind" are asserted about them-

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1. Principles, Introduction to 2nd. edition, p. xiv.

selves. The theory of types excludes the proposition under consideration from the range of its own significance, and thus places it in the next highest range. In this way the paradox is avoided.

In order to observe how this theory is applied to a specific paradox, let us examine how it solves the paradox of Epimenides, which asserts, in its simplest form, "I am lying." This assertion may be interpreted in this manner: "There is a proposition which I am affirming and which is false," or, in other words, I am asserting the truth of some value of the function "I assert p, and p is false." But "false" is an ambiguous term and to make it unambiguous the order of the proposition to which falsehood is ascribed must be stated. According to the theory of types, if p is a proposition of the nth order, then a proposition in which p occurs as an apparent variable is not of the nth order, but of a higher one. Hence the truth or falsehood which can belong to the statement "there is a proposition p which I am affirming and which has falsehood of the nth order" is a truth or falsehood which is of a higher order than the nth. Consequently, the statement of Epimenides does not come within its own range of significance, and, therefore, there is no contradiction. To make a statement with an apparent variable "of all of a kind" apply to itself is simply to make the statement meaningless or out of the range of its significance.

The theory of types adequately enables Russell and Whitehead to avoid the vicious-circle fallacies which resulted in the paradoxes, but it created new problems. The theory was not satisfied with merely solving the problems for which it was invented, but it applied itself in situations where it was not welcome and thus became troublesome. For example, it rendered it impossible to make a statement concerning truth, meaning, or logic and have the statement itself true or meaningful or logical within the meaning of its own terms. But of more importance to the mathematical logician was that the theory made it impossible to speak of all the properties of a term, which is a necessity in the development of mathematical logic. Thus it is necessary to make the theory of types less drastic.

To meet this new problem created by the theory of types, Russell and Whitehead introduced what is known as "the axiom of reducibility." Before we can state the axiom, we must give a definition of formal equivalence. It is defined as follows:[1] "two functions are 'formally equivalent' when they are satisfied by the same set of arguments," where an argument is one of the independent variables upon the value of which a function depends. The axiom of reducibility is stated as follows: "it is the assumption that, given any function $\phi\hat{x}$, there is a formally equivalent <u>predicative</u> function, i.e. there is a predicative function which is true

1. <u>Principia</u>, Vol. 1., p. 58.

when ϕx is true and false when ϕx is false."[1] For one variable, it is stated symbolically, as we have already observed in the previous chapter, in the following manner: $\vdash : (\exists f) : \phi x . \equiv_x . f!x$. For two variables, it is: $\vdash : (\exists f) : \phi(x,y) . \equiv_{x,y} . f!(x,y)$. This makes it possible for us to speak of all the properties of a function, since for every given function there is a formally equivalent predicative function of a higher order which asserts the truth or falsity of the given function. This function or proposition of a higher order, if its truth or falsity is to be asserted, must be judged by a function or proposition of still a higher order.[2]

This axiom is the weakest point in the Principia. At the time that it was put forward, the authors clearly indicated that they were not satisfied with it, but they accepted it and used it merely because it achieved certain desired results without dangerous consequences. It is neither demonstrable nor self-evident. It is included in its two forms as primitive propositions. We have already, under the section dealing with later modifications of the primitive propositions, seen how the axiom is discredited in the second addition of the Principia and how the results of the substitution of the two principles of Wittgenstein for the axiom are worked out in an appendix.

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1. Principia, Vol. 1, pp. 58-59.
2. By "function" we mean "propositional function" throughout this discussion.

Although the axiom of reducibility is not what the authors would like for it to be, the theory of types with this axiom enables them to work out the system of pure mathematics, with the exception of geometry which has not been subjected to this analysis and demonstration, by purely logical means from the logical calculus in <u>Principia.</u> The validity of their deductions, however, are yet to be considered. But we may assert at this time that their deductions do not involve vicious-circle contradictions. Thus we conclude that both the theory of types and the axiom of reducibility are justified until substitutes, which are more satisfactory, and will give the same or better results, are made. Wittgenstein's substitutes for the axiom may be more satisfactory in some respects, but Russell and Whitehead have been unable to secure all of the desired results from them, although they did get most of them in the second edition of <u>Principia.</u> At the time of the second edition, the authors were very dissatisfied with both the axiom of reducibility and the Wittgenstein substitutes.

Chapter VI

Russell's Proof of His Thesis

Russell's proof of his contention that all of pure mathematics can be deduced by purely logical means from the data given in his logical calculus is given in the Principles and this proof is given symbolic demonstration in Principia. In this chapter we shall be concerned with the proof as given, and we shall postpone our critical examination of the validity of the proof to the next chapter. The proof consists of two parts, namely, the giving of definitions of the various mathematical concepts--number, infinity, continuity, the various spaces of geometry, and motion--, and the establishment of certain mathematical existence-theorems. We shall consider the definitions first. They constitute the most important part of the proof.

A. Definitions of the Various Mathematical Concepts

The mathematical concepts considered here are: number, the infinitesimal, infinity, continuity, the spaces of geometry, and motion. Let us take these in the order enumerated and consider how Russell defined them in terms of the data in his logical calculus.

1. Number. Under this title in the Principles,[1] Russell attempts to show how the apparatus of his logical calculus, without new indefinables or new primitive propositions, is sufficient to establish the whole theory of cardinal integers as a special branch of symbolic logic. This he does by giving a definition of number in logical terms, showing how arithmetical addition and multiplication are dependent on logical addition, how both may be defined so that they are applicable equally to finite and infinite numbers, and how ratios and fractions are to be treated as explainable in logical terms. Also these mathematical deductions are tested critically by a consideration of the philosophical questions involved with the purpose of determining whether or not any new primitive ideas or postulates had crept in under cover. In all this he is confirmed that arithmetic contains no indefinables or indemonstrables other than those in general logic.

The first thing to be said about numbers is that they are applicable essentially to classes. While it is true that the individuals which make up a given finite number may be enumerated one by one without reference to a class-concept, all finite collections of individuals make up a class and the result is the number of a class. The individuals of infinite numbers cannot be counted one by one, and, consequently, an infinite number has to be defined by intention, that is "by some common property in virtue of

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1. Part II.

which they [the individuals or terms] form a class."[1] Thus for every class-concept there is a certain number of individuals denoted by it. Consequently, Russell considers "number" to be a property of a class, and he defines, nominally, the number of a class as "the class of all classes similar to the given class."[2] Two classes are said to be "similar" when they have the same number. This definition makes number the class-concept and not the class. A class-concept is a property, not a collection, by which a collectiøn is defined, and so number is really defined as "a common property of a set of similar classes and of nothing else."[3] This definition, it is claimed, permits the deduction of all the usual properties of finite and infinite numbers.

Now let us consider addition and multiplication as they are applicable to finite and infinite integers. Logical addition is said to be the fundamental kind and this is the same as disjunction. "The logical sum of two classes u and v may be defined in terms of the logical product of two propositions, as the class of terms belonging to every class in which both u and v are contained."[4]

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1. Principles, p. 113.
2. Ibid, p. 115.
3. Ibid, p. 115.
4. Ibid, p. 117. Consider: "If p and q are propositions, their logical sum is the proposition 'p or q,' and if u and v are classes, their logical sum is the class 'u or v,' i.e. the class to which belongs every term which either belongs to u or v." This explains the meaning of "logical sum."

This definition is extended to a class of classes, whether finite or infinite. "If k be a class of classes no two of which have any common terms, then the arithmetical sum of the numbers of the various classes of k is the number of terms in the logical sum[1] of k."[2] This is a general definition which applies equally to finite and infinite numbers.

The general definition of multiplication is stated as follows: "Let k be a class of classes, no two of which have any term in common. From which is called the multiplicative class of k, i.e. the class each of whose terms is a class formed by choosing one and only one term from each of the classes belonging to k. Then the number of terms in the multiplicative class of k is the product of all the numbers of the various classes composing k."[3] This definition also is applicable to both finite and infinite numbers.

At this point we might as well give the definitions of infinite and finite numbers. The class u is said to be an infinite class when it is possible to take away one term from u and leave a class u' similar to u. When this is impossible, the class is said to be finite. So it follows

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1. The logical sum of k means the logical sum of the classes composing k.
2. Principles, p. 118.
3. Principles, p. 119. This definition is accredited to A. N. Whitehead. See American Journal of Mathematics, October, 1902, for the article in which Whitehead presents the definition.

from these definitions that "the numbers of finite classes other than the null-class are altered by subtracting [or adding] 1, while those of infinite classes are unaltered by this operation."[1]

Peano developed the theory of finite numbers by introducing three indefinables and five primitive propositions which were necessary only for arithmetic. His three indefinables were: "0," "finite integer," and "successor of." He assumes that "successor of" means that every number has one and only one successor and that "successor" means "immediate successor." The five primitive propositions were: (1). 0 is a number. (2). If a is a number, its successor is a number. (3). If two numbers have the same successor, the two numbers are identical. (4). 0 is not the successor of any number. (5) If s be a class which includes 0 and the successor of every number included in s, then every number is included in s. With these he developed arithmetic, but this made arithmetic have its own indefinables and primitive propositions and thus arithmetic could not be rightly regarded as a branch of symbolic logic.

Russell, in the Principles, proved that Peano's new indefinables and new primitive propositions for arithmetic were not necessary. This he did by defining a class which satisfies all of the five primitive propositions of Peano and no more.[2] His definition is as follows:

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1. Principles, p. 121.
2. Ibid, p. 127.

> The class of classes satisfying his [Peano's] axioms is the same as the class of classes whose cardinal number is a_0....a_0 is the class of classes u each of which is the domain of some one-one relation R (the relation of a term to its successor) which is such that there is at least one term which succeeds no other term; every term which succeeds has a successor, and u is contained in any class s which contains a term of u having no predecessors, and also contains the successor of every term of u which belongs to s.

It is claimed that of every such class all the propositions of arithmetic of finite numbers can be proved.

But a simpler and more logically correct method of arriving at the same result as the above definition is the following:[1]

> (1). 0 is the class of classes whose only member is the null-class. (2). A number is the class of all classes similar to any one of themselves. (3). 1 is the class of all classes which are not null and are such that, if x belongs to the class, the class without x is the null-class; or such that, if x and y belong to the class, then x and y are identical. (4). Having shown that if two classes are similar, and a class of one term be added to each, the sums are similar, we define that, if n be a number, n+1 is the number resulting from adding a unit to a class of n terms. (5). Finite numbers are those belonging to every class s to which belongs 0, and to which n+1 belongs if n belongs.

This gives a complete definition of the finite numbers, and it is seen that finite numbers do not result from counting as is commonly supposed.

Russell regards ratios as relations of finite integers and fractions as relations between the divisibilities of aggregates. An aggregate is one type of

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1. Principles, p. 128.

wholes, the other type being a unity. An aggregate (manifold) is the whole formed of the terms of a collection. Such a whole is completely specified by the specification of all of its constituents. Unities, as wholes, are not specified by their terms. The former is of the greater use in mathematics. The type of "whole" known as "unity" is always a proposition. It has analytical parts but it does not have parts in the sense of constituents. Infinite unities may be logically possible but they do not appear in human knowledge. On the other hand, infinite aggregates are admitted. But infinite aggregates are of finite complexity. Nevertheless, if an infinite aggregate be divided, there must be at least one part which remains an infinite aggfegate. Thus fractions are relations between the divisibilities of wholes (aggregates). These divisibilities are magnitudes and a magnitude is defined as "whatever is greater or less than something else."[1]

This completes the most important points in Russell's treatment of the theory of cardinal numbers in the Principles. This same theory is demonstrated in Principia, and in Introduction to Mathematical Philosophy (1919) he had made no fundamental change in this treatment. Also in the introduction to the second edition of the Principles (1938) he points out the changes which have developed in his thought on the subject since 1903, and he mentions no changes in his treatment of the theory of numbers. However, during this time he did change from time to time the prim-

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1. Principles, p. 194.

itive ideas and propositions upon which this theory of number is based, as we have indicated elsewhere. But this was largely a matter of defining one idea in terms of another and of demonstrating one proposition by others in attempts to reduce the number of ideas and propositions remaining undefined and undemonstrated. These changes seldom affected the system deduced from the primitive ideas and propositions.

2. The Infinitesimal, Infinity, and Continuity. These subjects are dealt with at length in the *Principles* and in *Principia*, but the fundamental conclusions concerning them are presented in a summary fashion in the article on "Recent Work on the Principles of Mathematics,"[1] which may be regarded as an abstract of the *Principles*.

Zeno may be regarded as the founder of the problem of infinity. He discovered the problem in his consideration of the race between Achilles and the tortoise, and the flight of the arrow to its target, which is known to every student of philosophy. Zeno's problem was really three, namely, the problem of the infinitesimal, the infinite, and continuity. Practically every philosopher from the time of Zeno to our own day has dealt with these problems, but it was not until the mid-nineteenth century that any real progress was made. Russell bases his treatment of the subjects on the mathematical work of Weierstrass,

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1. *International Monthly*, Vol. 4 (1901), pp. 88ff.

who solved the problem of the infinitesimal, and Dedekind, who began and Cantor, who completed the solutions to the problems of infinity and continuity.

Before we go into the solutions of these particular problems, it is necessary for us to present in very brief form Russell's proof, which is based on the work of men like Weierstrass, Dedekind, and Cantor, that these problems of the infinitesimal, the infinite, and continuity are not concerned with quantity as it has been traditionally supposed, and as it is still assumed in practically all dictionaries. It is this proof that makes the concepts belong in the field of logic and explainable in logical terms. And, as we shall see, it was only when they were divorced from quantity that solutions of the many problems and antinomies involved in the matter were possible. This proof concerns the nature of quantity and order. We shall consider quantity first.

Russell arrives at a definition of quantity by a purely logical process. First, he asserts that there is a certain pair of indefinable relations called "greater" and "less."[1] These relations are shown to be asymmetrical and transitive and incompatible with each other. When one holds between A and B the other holds between B and A, and consequently each is the converse of the other. The terms which are capable of these two relations are called

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1. The relations "greater" and "less" are introduced as indefinables in the Principles. These are not included in his logical calculus, and they appear to be a weak point in his proof. But in Principia they are defined.

"magnitudes." Each magnitude has a relationship to some concept which is expressed as "a magnitude of that concept." Two magnitudes are said to be similar or of the same kind when they have this relation to the same concept. "Quantity" is said to be a magnitude particularized by a temporal, spatial, or spatio-temporal position, or "when, being a relation, it can be particularized by taking into consideration a pair of terms between which it holds."[1]

From magnitudes mathematicians have deduced the concept of infinity since it appears that of some kinds of magnitudes, for example, ratios, distances in space and time, there is a magnitude greater than any given magnitude. This magnitude is called infinity, and it is associated with quantity since quantity has been usually regarded as that from which magnitude has been deduced. Also philosophers have declared that every well-defined series of terms must have a last term. From these two conclusions the antinomies usually associated with the problem of infinity have arisen.

This problem of infinity, Russell asserts, "is not properly a quantitative problem, but rather one concerning order. It is only because our magnitudes form a series having no last term that the problem arises: the fact that series is composed of magnitudes is wholly ir-

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1. Principles, p. 167.

relevant."[1] The same is said of the infinitesimal. The problem of the infinitesimal is stated in this manner: for every given magnitude, there is a magnitude less than the given one, and also philosophers have asserted that every well-ordered series must have a first term as well as a last one. But again this has no reference to quantity and only to magnitudes in the sense that it deals with a series composed of magnitudes. Continuity is defined in this manner: "It applies to series (and only to series) whenever these are such that there is a term between any two given terms. Whatever is not a series, or composed of series, or whatever is a series not fulfilling the above condition is discontinuous."[2] Thus we see that infinity, the infinitesimal, and continuity have to do with series, which means that they are related to order and not to quantity as it has been regarded traditionally.

We must now consider order. We may sum up in a very simple conclusion the results of a long complicated discussion on this matter. The conclusion is: "The minimum ordinal proposition [proposition of order], which can always be made wherever there is an order at all, is of the form: 'y is between x and z.'"[3] This means that "there is some asymmetrical transitive relation which holds between x and y and between y and z."[3] Furthermore, it

1. Principles, p. 189.
2. Ibid, p. 193.
3. Ibid, p. 217.

is said that all order involves asymmetrical transitive relations, and that every series as such is open. "Asymmetrical transitive relations" and "open series" require explanation.

An "asymmetrical transitive relation" is a relation which possesses the properties of asymmetry and of transitiveness, and this type of relation is declared to be that which generates order or series. This places the essence of order in the relation among the members of a class and not in the class of terms to be ordered. An asymmetrical relation is a relation of the type xRy, which excludes the possibility of yRx. For example, the relation of "preceding." If x has the relation of "preceding" to y, it is not possible for y to have the relation of "preceding" to x in the same series. This is called an asymmetrical relation. A transitive relation is of this sort: if x precedes y and y precedes z, then x precedes z. If a relation possessing both of these properties, namely, asymmetry and transitiveness, exists among the terms of a class, then order exists or the terms are arranged in a series.

In the Principles, the above is regarded as an adequate explanation of order, but later[1] Russell finds it necessary to make explicit a third property of a relation if it is to generate order or a series. This third pro-

1. Principia, Vol. 11, p. 513, and Introduction to Mathematical Philosophy, p. 32.

perty is called "connexity." This property had been associated with the series itself and not with the relation in the *Principles*. "Given any two terms of the class which is to be ordered, there must be one which precedes and the other which follows. For example, of any two integers, or fractions, or real numbers, one is smaller and the other greater; but of any two complex numbers this is not true. Of any two moments in time, one must be earlier than the other; but of events, which may be simultaneous, this cannot be said. Of two points on a line, one must be to the left of the other."[1] A relation having this property is called "connected." Thus a relatibn is "serial" when it is asymmetrical, transitive, and connected. A "serial" relation is the same as a "series." There are many ways of generating series, but all these ways depend upon the finding or construction of an asymmetrical, transitive, connected relation.

A series is said to be "open" when it does not have a beginning, or when it has a beginning that is not arbitrary. It is closed when it has an arbitrary beginning. This means that it has a first term. Every closed series can be opendd and every open series closed mathematically. Yet, in regard to the nature of the generating relation, there is a genuine philosophical difference between the two, but this is of little, if any, mathematical importance.

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1. *Introduction to Mathematical Philosophy*, p. 32.

Thus, instead of being concerned with quantity, the infinitesimal, the infinite, and continuity are concerned with the realms of order and number. Order and number have been defined in purely logical terms without the introduction of additional indefinables and primitive propositions. It remains to be seen how these concepts can be defined and how the problems involved in them can bé solved.

The Infinitesimal

The infinitesimal has been regarded as very important to mathematics. The Greeks invented the idea in their distinction between a circle and a polygon. They asserted that a circle is merely a polygon with a large number of very small equal sides. After Leibniz invented the infinitesimal calculus, the notion of the infinitesimal became one of the fundamental notions of all higher mathematics. This calculus required continuity and continuity was regarded as possible only on the basis of the infinitesimal or the infinitely little elements. But no one was able to discover what the infinitesimal or the infinitely little really was. It was evident that it was not zero. But when the infinitesimals were added together, they seemed to make up a finite whole. And no one could discover a fraction which was not finite. So a stalemate developed in regard to the problem until the time of Weierstrass about the middle of the last century. Weierstrass discovered that the infinitesimal was not needed at all, and that everything that had been regarded as dependent upon it could be accomplished without it. Consequently, the whole idea of the infinitesimal was abandoned. This view is demonstrated to be correct in the Principles and it is generally accepted today. Russell shows that the term "infinitesimal" is relative and that it is not capable of being more except in regard to magnitudes that are divisibilities, or divisibilities of wholes which are infinite in the absolute sense. But where it has absolute meaning, it is shown to be nothing more than finitude. Infinity and continuity are shown tø be completely independnet of this concept.

The Probołem of the Infinite

Traditionally, it was assumed that infinite numbers and the mathematically infinite generally were self-contradictory. But it was obvious that there were infinites, the number of numbers for example. The contradictions seemed inevitable and philosophy seemed to have wandered into a "cul-de-sac." This lead to Kant's antinomies and to much of Hegel's dialectical method. But Russell declared in 1901 that "all the ancient and respectable contradictions in the notion of the infinite have been once for all disposed of."[1] In this statement

1. "Recent Work on the Principles of Mathematics," Int. Monthly, Vol. 4, p. 92.

he had in mind the solution offered by Dedekind and Cantor. He accepted their solution at that time and he has never rejected it. Subsequently it has been accepted by many mathematicians and philosophers.

Let us examine the procedure of Dedekind and Cantor in their solution of the problem of infinity. In the first place, they asked the question: "What is infinity?" Russell says that this question had never been asked intelligently. Then they found a perfectly precise definition of infinite number or an infinite collection of things by examining the supposed contradictions in the notion. Cantor strictly examined the supposed proofs of pairs of contradictory propositions in which both sides of the contradiction were regarded as demonstrable. He discovered that all proofs opposed to infinity involved a certain maxim which appeared to be obviously true, but which always resulted in destructive consequences to almost all parts of mathematics. On the other hand, the proofs which were favorable to infinity never involved any detrimental principles. Thus he concluded that common sense had been leading philosophers and all others astray on this one principle involved in all of the proofs opposed to infinity.

The diabolical principle in question is that if one collection is a part of another, the part has fewer terms than the collection of which it is a part. This principle is true of finite numbers, but in infinite numbers it invariably lead to disastrous results in all of mathematics and philosophy. Cantor decided that the principle was false in regard to infinite numbers and so he abandoned it and assumed that a collection which is a part of another collection, when the collections are infinite, has just as many terms as the collection of which it is a part.

Consequently, we get this definition: "A collection of terms is infinite when it contains as parts other collections which have just as many terms as it has. If you can take away some of the terms of a collection, without diminishing the number of terms, then there are an infinite number of terms in the collection. For example, there are just as many even numbers as there are numbers altogether, since every number can be doubled,"[1] and the double of every number is an even number. Thus the number of finite numbers is infinite.

Ordinal numbers are arrived at by counting, that is to say that counting gives numbers in an order or a series. When there is only a finite number of terms, we can count them in any order we like. Ordinal numbers depend upon the number of terms and the way in which they are arranged. True infinite numbers are cardinal. Cardinals are not determined by counting them, since they do not tell us the number of terms a collection has. They tell us whether two collections have a one-one relation of terms, that is whether two collections have the same number of

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1. "Recent Work on the Principles of Mathematics," Int. Monthly, Vol. 4, p. 95.

terms, or whether one has more or less than another.

The numbers of infinite collections are defined by the following method: "If every term of a collection can be hooked on to a number, and all the finite numbers are used once, and only once, each in the process, then our collection must have just as many terms as there are finite numbers."[1]

We are told that there are infinitely more infinite numbers than finite ones. Thus all infinite numbers are not equal. Russell says that:[2]

> there are probably more points in space and more moments in time than there are finite numbers.[3] There are exactly as many fractions as whole numbers, although there are an infinite number of fractions between any two whole numbers. But there are more irrational numbers than there are whole numbers or fractions. There are probably exactly as many points on a line a millionth of an inch long as in the whole of infinite space.

In the article on "Recent Work on the Principles of Mathematics," Russell parted company with Cantor on the matter of the existence (or subsistence) of "a greatest of all infnite numbers, which is the number of things altogether, of every kind and sort." Cantor had said that there is no such number, but Russell asserted that there is and he promised to prove it in a later book. Evidently he had in mind proving it in the Principles, which was almost

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1. "Recent Work on the Principles of Mathematics," Int. Monthly, Vol. 4, p. 95.
2. Ibid, p. 95.
3. Later Russell substituted logical constructions composed of events for points of space, instants of time, and particles of matter. See Principles,(2nd. ed.),p.xi.

ready for publication at the time the article was written. But in the Principles,[1] after having carefully examined Cantor's work, Russell says: "The only solution I can suggest is, to accept the conclusion that there is no greatest number..." In 1901 he said that if a greatest infinite number was not accepted all of the old contradictions involved in the infinite would return. But in the Principles and later works the theory of types is used as the way out of the contradictions which returned home after the abolition of the notion of a greatest of all infinite numbers.

In addition to the definition of infinity given above, Russell gives another in the Principles. The other one states that an infinite number is that which cannot be reached by mathematical induction starting from 1. This is called an ordinal definition of infinite number, while the one given above, is called a cardinal definition. The cardinal definition is the most important one.

Continuity

The solution of the problem of infinity made it possible to solve the problem of continuity. This solution is due to Cantor also. He has been able to give a perfectly precise definition of continuity and to show that there are no contradictions involved in the notion as defined. Russell accepts this treatment and shows how it

1. p. 368.

is part of the pure mathematics deduced from formal logic by showing that it has to do exclusively with order and not with quantity,as we have already shown. Cantor gave two definitions of continuity, but the first one is not purely ordinal. In at least two points, as Russell points out, it demands some reference to numbers or numerically measurable magnitudes. His later definition, which is shown by Russell to be purely ordinal, is the one that we are interested in.

Before we can give the definition itself it is necessary to give some auxillary conceptions so that the definition may be understandable when it is given. First of all we must consider the type of order exhibited by the series of rational numbers which are greater than zero and less than one. This series has three peculiarities which define it. (1). It is"denumerable," that is its terms can be arranged so that they have a one-one relation with the successive integers; (2). it has neither a first nor a last term; and (3). between any two terms there is always a third. Any series that possesses these three characteristics is said to be of the type of order η.

We now have to consider what Cantor calls a "fundamental series." We need to consider only the ascending fundamental series. By this is meant a series of which the terms have the type of order η defined above. Such a series S is said to have a limit in η, if there is a term in η which is the first after all the terms of S. Then we say that any manifold (aggregate) is perfect if all the fundamental series contained in it have a limit in it, and if all its terms are limits of fundamental series contained in it.

With these conceptions given, the definition of the continuum is as follows: "A one-dimensional dontinuum M is a series which (1) is perfect, (2) contains within itself a denumerable series S of which there are terms between any two terms of M."[1] Cantor proves that any series M that satisfies this definition is ordinally similar to the number continuum, that is the real numbers from 0 to 1, both inclusive. What is meant by "real" numbers? We are told that "The series of real numbers, as ordinally defined, consists of the whole assemblage of rational and irrational numbers, the irrationals being defined as the limits of such series of rationals as have neither a rational nor an infinite limit."[2] However, Russell does not accept the existence of any irrationals as defined in this statement. He considers them as a form of rational numbers, but this does not concern the question at hand. That this definition of the continuum is satisfied by the series of real numbers is clear since it can easily be shown that the series is "perfect" and that

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1. *Principles*, p. 297.
2. *Ibid*, p. 270.

there is always at least one term of the series of rational numbers between any two real numbers.

This solution of the problem of continuity is of great significance in philosophy and mathematics. One of the results is the proof that geometry canbe divorced from the space in which we live and can be made compatible with the theory of number. This makes it possible for geometry to be deduced from the primitive ideas and propositions of the logical calculus.

Thus the problems of infinity, the infinitesimal, and continuity are solved. These solutions have been widely accepted by both mathematicians and philosophers during the past forty years. Concerning the solution to the problem of the infinite, which may be regarded as the key one of the three, Russell says:[1]

> The solution of the difficulties which formerly surrounded the mathematical infinite is probably the greatest achievement of which our age has to boast. Since the beginnings of Greek thought these difficulties have been known; in every age the finest intellects have vainly endeavoured to answer the apparently unanswerable questions that had been asked by Zeno the Eleatic. At last Georg Cantor has found the answer, and has conquered for the intellect a new and vast province which had been given over to Chaos and Night.

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1. Mysticism and Logic (1929), p. 64.

3. Space. This aspect of our subject has to do with the branch of mathematics known as geometry. Russell's second book was entitled "An Essay on the Foundations of Geometry" (1897). In addition to this Part VI of the Principles deals with the subject. In volume III of the first edition of Principia and again in the second edition a fourth volume of the work is promised in which geometry would be treated, but this additional volume has not been published. Consequently geometry has not been given as precise symbolic treatment as the other branches of mathematics, but some of the essentials of the subject are covered in the Principles. When the first book on geometry was published in 1897, Russell had not developed his logico-mathematical thesis. Consequently, the book is of little value for our purposes.

Until about a century ago there was only one geometry, namely, Euclidean, which was based on the common sense belief in the space in which we live. But during the nineteenth century several non-Euclidean geometries based on non-Euclidean spaces were developed. This raised serious questions concerning the nature of space. Russell concluded in his first work on the subject in 1897 that space must be regarded as ordinal and not quantitative. This was a step toward his later logico-mathematical thesis, and it helps in proving the thesis. The conclusion of his book on the Foundations of Geometry is the following: "Space, if it is to be free from contraditions, must be regarded exclusively as spatial order, as relations between unextended material atoms. Empty space, which arises, by an inevitable illusion, out of the spatial element in sense-perception, may be regarded, if we wish to retain it, as the bare principle of relativity, the bare logical possibility of relations between diverse things. In this sense, empty space is wholly conceptual; spatial order alone is immediately experienced."[1]

In the Principles,[2] Russell says: "As a branch of pure mathematics, Geometry is strictly deductive, indifferent to the choice of its premisses and to the question whether there exist (in the strict sense) such entities as its premisses define. Many different and even inconsistent sets of premisses lead to propositions which would be called geometrical, but all such sets have a common element. This element is wholly summed up by the statement that Geometry deals with series of more than one dimension." Thus the definition is: "Geometry is the study of series of two or more demensions."[3] Such series are found to arise whenever we have a series composed of terms which are series themselves.

By the abstract logical, method based on the logic

1. Foundations of Geometry (1897), pp. 197f.
2. Principles, p. 372.
3. Principles, p. 372.

relations, Russell shows how the several geometrical spaces may be defined and how the propositions of projective, descriptive, and metrical geometries may be proved without introducing any new primitive ideas or propositions. Let us consider the definitions of the several spaces. He gives a definition of projective space, which serves as a model for descriptive and metrical spaces, but he does not give the definitions of the latter two. The definition of projective space of three dimensions is as follows:

> A projective space of three dimensions is any class of entities such that there are at least two members of the class; between any two distinct members there is one and only one symmetrical aliorelative[1], which is connected, and is transitive so far as its being an aliorelative will permit, and has further properties to be enumerated shortly; whatever such aliorelative may be taken, there is a term of the projective space not belonging to the field of the said aliorelative, which field is wholly contained in the projective space, and is called, for shortness, a straight line, and is denoted by <u>ab</u>, if <u>a</u>, <u>b</u> be any two of its terms; every straight line which contains two terms contains at least one other term; if a, b, c be any three terms of the projective space, such that c does not belong to the class ab, then there is at least one term of the projective space not belonging to any class cx, where x is any term of ab; under the same circumstances, if a' be a term of bc, b' a term of ac, the classes of aa', bb' have a common part; if d be any term, other than a and b, of the class ab, and u, v any two terms such that d belongs to the class uv, but neither u nor v belong to the class ab, and if y be the only term of the common part of au and bv, z the only term of the common part of av and bu, x the only term of the common part of yz and ab, then x is not identical with d (under these circumstances it may be proved that the term x is independent of u and v, and is uniquely determined by a, b, d; hence x and d have a sym-

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1. An "aliorelative" is a relation which no term has to itself.

> metrical one-one relation which may be denoted, for brevity, by $xH_{ab}d$; if y, e be two further terms of the projective space, belonging to the class xd, and such that there are two terms g, h of the class xd for which we have $gH_{xd}h$ and $gH_{ye}h$, then we write for shortness $yQ_{xd}e$ to express this relation of the four terms x,d,y,e.); a projective space is such that the relation Q_{xd}, whatever terms of the space x and d may be, is transitive; and that, if a, b, c, d be any four distinct terms of one straight line, two and only two of the propositions $aQ_{bc}d$, $aQ_{bd}c$, $aQ_{cd}b$ will hold; from these properties of projective space it results that the terms of a line form a series; this series is continuous...; finally if a,b,c,d,e be any five terms of a projective space, there will be in the class <u>ae</u> at least one term x, and in the class <u>cd</u> at least one term y, such that x belongs to the class <u>by</u>.[1]

This is a rather long and complicated definition, but any purely formal definition would have to be. Any class of entities that fulfills this definition is said to be projective space. There is a whole class of projective spaces and this class has an infinite number of members, Russell says, and we know four such members, namely, arithemtical space, the projective space of descriptive geometry, the polar form of elliptic space, and the antipodal form of elliptic geometry.

In his treatment of space in the <u>Principles</u>, Russell concludes that space is composed of points and that the number of points must be equal to or less than the number of the continuum. This is known as the theory of absolute position. It holds that the relations with which

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1. <u>Principles,</u> pp. 430-431.

we have been concerned hold between spatial points, which essentially and timelessly have the relations which they do have. This is opposed to what is known as relative position, which holds that spatial relations have to do with material points which are capable of motion or change in their spatial relations. Russell shows how there is no logical arguments against space being composed of points. Hence he concludes that there is such a thing as absolute space. But later Whitehead persuaded Russell to abandon points of space along with instants of time, and particles of matter and to substitute for them logical constructions composed of events.[1]

4. Motion. In dealing with the problem of the infinite, Zeno was forced to conclude that there is no such thing as motion. The problem of motion, which is the basis of dynamics, baffled philosophers until the time of the work of Weierstrass. Russell presents a theory of motion which is defined in purely logical terms. But before we can give this theory it is necessary for us to say something about his theory of matter, since motion is usually regarded as concerned with particles of matter. But, if a definition of motion is to be derived from the logical calculus, motion must be divorced from material particles.

He sums up his conception of the nature of matter in 1903 as follows:[2]

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1. See the Introduction to the second edition of the Principles, p. xi. Also see Russell, Philosophy, (1927), pp. 276ff.
2. Principles, p. 468.

> Material unit is a class-concept, applicable to whatever has the following characteristics: (1) A simple material unit occupies a spatial point at any moment; two units cannot occupy the same point at the same moment, and one cannot occupy two points at the same moment. (2) Every material unit persists through time; its positions in space at any two moments may be the same or different; but if different, the positions at times intermediate between the two chosen must form a continuous series. (3) Two material units differ in the same immediate manner as two points or two colours; they agree in having the relation of inclusion in a class to the general to concept matter, or rather to the general concept material unit.

Thus matter seems to be a collective name for the constituents which make it up. It is the class-concept of the class of bits of matter, or, according to the later revision due to Whitehead, the class of logical constructions composed of events, in the same sense that man is a class concept for the class of men.

After the nature of matter is set forth in the above manner, we have an abstract statement of matter as it is used in rational dynamics, and it is motion as concerned in rational dynamics that we are interested in. First, time and space are replaced by a one-dimensional and n-dimensional series respectively. Then it is evident that "the only relevant function of a material point [or a logical construction composed of an event as he would have said later] is to establish a correlation between all moments of time and some points of space, and that this correlation is many-one."[1] The actual material point loses its im-

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1. Principles, p. 486.

portance since the correlation is given. Thus the material point is replaced by a many-one relation whose "domain is a certain one-dimensional series, and whose converse domain is contained in a certain three-dimensional series."[1]

Now with the material particles, or logical constructions composed of events, replaced by many-one relations of all times to some places, or "of all terms of a continuous one-dimensional series <u>t</u> to some terms of a continuous three-dimensional series <u>s</u>," we may state Russell's theory of motion.[2] He says:

> Motion consists broadly in the correlation of different terms of <u>t</u> with different terms of <u>s</u>. A relation R which has a single term of <u>s</u> for its converse domain corresponds to a material particle which is at rest throughout all time. A relation R which correlates all the terms of <u>t</u> in a certain interval with a single term of <u>s</u> corresponds to a material particle which is at rest throughout the interval, with the possible exclusion of its end-terms (if any), which may be terms of transition between rest and motion. A time of momentary rest is given by any term for which the differential coefficient of the motion is zero. The motion is continuous if the correlating relation R defines a continuous function. It is to be taken as part of the definition of motion that it is continuous, and that further it has first and second differential coefficients.

In this theory of motion, which is based on the denial of the infinitesimal, motion consists merely in an entity's occupation of different points at different times, subject to continuity. There is no such thing as a state of motion. There is no transition from point to point, and no consecutive positions or moments since there is no such

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1. <u>Principles</u>, p. 468.
2. <u>Ibid</u>, p. 437.

thing as the next point or the next moment. This is due to the fact that between any two points or moments there must be a third. Consequently velocity and acceleration as physical facts are rejected. All that we can say is that an entity is at one point in a series at one moment and at another point at another moment. We can say nothing about what happens in the interval. This conclusion is made imperative by the work of Weierstrass in abolishing the infinitesimal.

Thus Russell defined all of the fundamental mathematical concepts, namely, number, infinity, continuity, space, and motion, in terms of his logical calculus. These concepts and the theories involved in them give us the fundamental definitions necessary for arithmetic, geometry, and dynamics. These definitions are always the definition of a class or of a single member of a unit class. This is necessary since the only way of giving a definition of these terms is to give a propositional function which the object or objects to be defined is to satisfy.

B. The Existence Theorems

In addition to these definitions, we need the existence theorems of mathematics, which are the proofs that these various classes defined are not null, in order to show that all of mathematics can be deduced from the logical calculus. We have already given some of these along with the consideration of the definitions. But let us state the most important of them in summary fashion at this point. Almost all of them are obtained from arithmetic. The existence of zero is derived from the fact the null-class is included in it, and 1 is proved to exist by the fact that zero is a unit class, since the null-class is its only member. The existence of all the finite numbers is proved by the fact that if n is a finite number, n+1 is the number of numbers from 0 to n (both included). Hence the existence of α_0, the smallest of the infinite cardinal numbers, follows from the class of finite cardinal numbers themselves, since the number of finite numbers is not finite. It is clear that the number of finite numbers is infinite since the number of even finite numbers is equal to the number of all finite numbers (By definition, the infinite is that which contains a part which has just as many terms as that of which it is a part.). In like manner the existence of ω, the smallest of the infinite ordinals, follows from the series of finite cardinals in order of magnitude. The existence of η, the type of endless compact[1] denumerable series, follows from the definition of rational numbers and of their order of magnitude. And from the segments of the series of rational numbers the existence of real numbers[2] and of θ, the type of continuous series,

1. A compact series is one in which there is a term between any two terms.
2. Russell regards the real numbers as a part of the rational numbers.

follow. From the fact that the number of well-ordered types from 0 to α is $\alpha + 1$, and from the fact that "if <u>u</u> be a class of well-ordered types having no maximum, the series of all types not greater than every <u>u</u> is itself of a type greater than every <u>u</u>" the existence of the terms of well-ordered types are proved. The existence of the class of Euclidean spaces of any number of dimensions is proved from the existence of θ by the definition of complex numbers. Also Russell proves the existence of projective, non-Euclidean descriptive, and metrical spaces, but it is not necessary for us to consider these proofs here. Lastly, the class of dynamical worlds is proved to exist by "correlating some of the points of a space with all the terms of a continuous series."[1]

Throughout the <u>Principles</u> all of these definitions and existence-theorems are derived without the introduction of any entity that is not definable in terms of the indefinables of the logical calculus or demonstrable in terms of the primitive propositions. He concludes that "the purely logical nature of mathematics is established throughout." We will not pass judgment on this conclusion until we have subjected the more important parts of his proof to a critical examination in the next chapter.

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1. All of these existence-theorems are collected in the <u>Principles</u>, pp. 497-498.

Chapter VII

Examination of Russell's Proof of His Thesis

The purpose of this chapter is to determine the validity of Russell's proof of his logico-mathematical thesis. We have already seen how he has defined the various fundamental mathematical concepts and deduced the fundamental existence-theorems in the Principles. These facts are cogent, but Russell and Whitehead were not satisfied merely with them. They sought to give precise mathematical demonstration of the deduction of all of pure mathematics from their primitive ideas and propositions in the Principia. However, as we have pointed out, the volume on geometry has not been published. In Principia the conclusions which had been stated in the Principles, with some modifications, are deduced in symbolic form step by step in minute detail. In our critical examination of the proof presented for the establishment of the thesis, we shall use Principia as our source material. Jørgensen,[1] in his magnificant work on formal logic, gives an excellent critical examintion of the Russell-Whitehead proof of their logico-mathematical thesis. In our discussion, we shall rely heavily upon this work since we believe it to be the best and soundest critical examination of the thesis of our philosopher to date.

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1. Jørgen Jørgensen, A Treatise of Formal Logic (Copenhagen: Levin and Munksgaard Publishers, 1931), Vol. III, pp.59-200.

Our purpose is to determine whether or not the proof of the thesis is valid. To do this it is not necessary for us to examine the deduction of all the properties of pure mathematics from the logical calculus. If we did this we have the Principia reproduced here. We need to consider only some of the more fundamental mathematical concepts, and we choose to examine how the theory of natural numbers, from which the theories of qualified, rational, real and complex numbers can be derived, and how the theory of infinite numbers can be deduced from the primitive ideas and propositions of the Principia. This will firmly establish the deduction of both finite and infinite arithmetic, that is the arithmetic of both finite and infinite numbers. We will have to accept the proof of the deduction of geometry given in the Principles as probably true since it has not been given symbolic demonstration. We have already pointed out how the primitive ideas and propositions of Principia differ from those of the Principles, but these differences, as we have said elsewhere, merely consist of the defining of one term in terms of others and of demonstrating one proposition by means of the others. Either list is acknowledged to be arbitrary. Such differences between the components of the calculus do not affect the validity of the proof of the thesis.

A. The Theory of Natural Numbers

Mathematicians regard the series of natural numbers as the basis of the whole structure of mathematics and this series is considered to be known. It is understood to be the infnite series of finite absolute integers expressed in Arabic symbols as 0,1, 2, 3.......n. Each of these figures can be regarded either as a number of objects or terms or as a numeral indicating a spedific place in the series. The former is known as a cardinal number and the latter as an ordinal. The cardinals are the most important and most fundamental since the ordinals cannot be different from what they are if they are arranged according to magnitude and if we know the meaning of being "greater than" or "less than" by one unit. Each ordinal is determined by its relation of a unit magnitude to its predecessor and its successor. Therefore, in order to give the natural series of numbers we must know every item in it and the relation that exists between everyone of them and the one immediately before and the one immediately after. Consequently we have to know the cardinal numbers and be able to arrange them so as to form the series of natural numbers before we can define any arbitrary ordinal number. From this we conclude that cardinal numbers are more fundamental than ordinals.

Our first problem then is to determine what a cardinal number is. Many mathematicians have said that this is an impossible task, but some mathematicians and logicians have insisted on giving a definition. As we have seen, Russell, in the <u>Principles</u>, defines the number of a class as "the class of all classes similar to a given class."[1] In the <u>Principia</u>, this definition is given as follows:

*100.02. $NC = D'Nc$ Df. (definition)

where NC is the class of cardinal numbers and where Nc is defined as under:

*100.1 $Nc = \overrightarrow{sm}$ Df.

which means that Nc is the relation "similar" between classes. D is the field of the relation. Thus the definition reads: The class of cardinal numbers is the field of the relation between similar classes, or, in other words, the class of cardinal numbers is the class of classes which are similar to each other. If α be a class, its cardinal number ($Nc'\alpha$) will be the class of all classes similar to it, which means that the cardinal number of a class is the class of all classes similar to a given class.

The problem now is to determine whether this definition is logically correct and whether it fits the mathematical meaning of cardinal numbers and nothing else. The latter is not difficult to prove since it asserts that the class of cardinal numbers is identical with cardinal numbers, which is a pure tautology and perfectly obvious. But objections have been raised conserning the logical correctness of this definition. Let us consider some of these.[1]

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1. These objections are listed and summarized by Jørgensen.

1. F. Hausdorft[1] objects to this definition on the basis that the concept "class of all classes" is invalid since it leads to the "Russell Paradox," that is that "the class of all classes" includes itself as a member of itself and therefore is guilty of the vicious-circle fallacy. As we have already seen Russell resolves this difficulty with his theory of types and the axiom of reducibility.

2. A second objection to this definition of cardinal numbers is raised by J. Mollerup.[2] This objection says that the definition is self-contradictory in that it makes the number one the aggregate (or manifold) of all things. One, according to the definition, is the class of all unit classes, which includes all individual things. But this does not constitute a contradiction. Even if there were an infinite number of unit classes, one is not said to be equal to infinity. One, according to the definition, is identical with the class of all unit classes, but this is not identical with the number of all unit classes. It is identical with the class of all classes which are similar to the class of all unit classes. Mollerup's difficulty seems to be the confusion of "class of all classes similar to a given class" with the "number of all classes similar to a given class."

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1. <u>Grundzüge der Menglehre</u> (Leipzig, 1914), pp. 46,450.
2. "<u>Die Definition des Mengenbegrieffs</u>" in <u>Math. Ann.</u>, Vol. 64 (1927), p. 231.

3. Also it has been objected[1] that the cardinal number of a class cannot be regarded as the aggregation of all aggregates equivalent to the class, since this is not known. But there is no need of knowing all the members of a class in order for the class to be regarded as well-known and well-defined. All that is required is a way of knowing whether or not any given object is a member of the class. Therefore this objection also is over ruled.

The series of natural numbers is characterized by the fact that every term in it can be obtained by the addition of 1 to the term immediately preceding it, and it is an accepted fact that all of the known mathematical properties of the natural numbers can be deduced from the series of natural numbers. Then all that is necessary to prove that the theory of natural numbers can be derived exclusively from the logical calculus is to show how 0, 1, and the addition of 1 to a given number can be defined in purely logical terms. From these three definitions we can construct every term in the series. Then we have to determine whether this series has the characteristic properties of the series of natural numbers.

First, let us consider the definitions of 0 and 1. In the <u>Principles</u>[2], 0 is defined as "the class of

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1. Weber-Wellstein; <u>Encyclopädie der Elementar-Mathematik</u> (Leipzig, 1909), <u>Vol. I., p. 13</u>.
2. p. 128.

classes whose only member is the null-class." In the Principia this definition is stated symbolically:

$*54.01.\ 0 = \iota`\Lambda$

where Λ means the null-class and $\iota`\Lambda$ means the class whose only member is the null class. We have the same definition of 1 in both the Principles[1] and in Principia, which is stated symbolically in the latter in this manner:

$*52.01.\ 1 = \hat{\alpha}\{(\exists x).\alpha = \iota`x\}$

which means that 1 is the class of all unit classes, or, as it is stated in the Principles: "1 is the class of all classes which are not null and are such that, if x belongs to the class, the class without x is the null-class; or such that, if x and y belong to the class, then x and y are identical.

These definitions are in accord with the definition of cardinal numbers since it can be proved that:

$*101.11.\ \vdash .\ 0 \varepsilon NC$ and

$*101.21.\ \vdash .\ 1 \varepsilon NC$

These respectively: "It is asserted that 0 is included in the class of cardinal numbers," and "It is asserted that 1 is included in the class of cardinal numbers."[2]

Our next problem is to determine whether or not these definitions are logically unassailable. It has been maintained by many that any definition of 0 or 1 must be

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1. P. 128.
2. For the demonstration of the proof see these numbers in the Principia.

guilty of the vicious circle fallacy.[1] Our task is to determine whether the above definitions of Russell and Whitehead are guilty of circularity. To do this we must determine whether or not the idea of the null-class presupposes the idea of 0 and the idea of a unit class presupposes the idea of 1.

The null-class, which is symbolized as Λ, is defined as the class which has no members, or as the negation of the universe of discourse, which is symbolized $-V$, which is the class of all objects identical with themselves. These definitions are expressed in this way in the Principia:

$*24.01.\quad V = \hat{x}(x = x)\qquad Df.$

$*24.02.\quad \Lambda = -V\qquad Df.$

in which $-V$, according to $*22.04$, is equal to $\hat{x}(x \sim\epsilon V)$, which means the class of objects not included in the class of the universe of discourse. In defining the null-class, therefore, only three things have to be known, namely, (1). what is meant by universe of discourse, (2). what is meant by ϵ, the relation of individual to the class of which it is a member, and (3). what is meant by negation. In no way has the idea of 0 been presupposed in this definition of the null-class. Consequently the charge of the vicious-circle fallacy in the definition of 0 is unwarranted.

We now have to consider whether or not the idea of a unit class presupposes the idea of 1. A unit class is

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1. See H. Poincare, Science et Methode, (Paris, 1916), p.168f. and P. Natrop, Die logischen Grundlagen der exakten Wissenschaften (Leipzig u. Berlin, 1910), pp. 112 ff.

"a" is indefinite. "One" is definite. "a" is a variable with indefinite meaning. "One" is a constant with definite meaning. This meaning of "one" is determinable by means of the indefinite "a," but the meaning of the indefinite "a" is not determinable by the numeral one, since a single constant can never define or explain in any way the meaning of a variable. Therefore the indefinite article "a" is logically more fundamental than the numeral "one" and the former does not in any way presuppose the latter. Thus Russell's and Whitehead's definition of 1 is validated.

Since we have shown that the definitions of 0 and 1 are logically correct, that is we have shown that they contain no logical fallacies, we must now raise the question as to whether these objects (0 and 1) as defined are identical with what mathematicians dall 0 and 1. We can determine this only by showing whether or not they have the qualities which 0 and 1 have in arithmetic. These qualities are expressed in arithmetic in the following manner:

$$0+0=0$$
$$0+1=1+0=1$$
$$1+1=2$$
$$0\times 0=0$$
$$0\times 1=1\times 0=0$$
$$1\times 1=1$$
$$1\times 2=1+1$$

In the <u>Principia</u>, these qualities are proved to belong to the logically defined 0 and 1 by the following propositions:

*110.64. $\vdash . 0 +_c 0 = 0$

where the suffix $_c$ following the addition sign indicates that the two objects added are cardinals of assigned types.

*110.641. $\vdash . 1 +_c 0 = 0 +_c 1 = 1$

*110.643. $\vdash . 1 +_c 1 = 2$

*113.6. $\vdash . \mathrm{Nc}`\alpha \times_c 0 = 0$ and

*113.27. $\vdash . \mu \times_c \nu = \nu \times_c \mu$

where μ and ν are two cardinal numbers.

*113.66. $\vdash . \mu \times_c 2 = \mu +_c \mu$

But we have gone a little ahead of ourselves. These formulae presuppose the definitions of addition and multiplication. We have seen how these were defined logically in the Principles. In the Principia these definitions are given symbolically as follows:

*110.02. $\mu +_c \nu = \hat{\xi}\{(\exists \alpha, \beta) . \mu = N_0c`\alpha . \nu = N_0c`\beta . \xi \, \mathrm{sm} \, (\alpha + \beta)\}$ Df.

*113.03. $\mu \times_c \nu = \hat{\xi}\{(\exists \alpha, \beta) . \mu = N_0c`\alpha . \nu = N_0c`\alpha . \xi \, \mathrm{sm} \, (\alpha \times \beta)\}$ Df.

The sum and the product of two cardinal numbers defined in this manner can be proved to be themselves cardinal numbers:

*110.42. $\vdash . \mu +_c \nu \in \mathrm{NC}$

*113.23. $\vdash . \mu \times_c \nu \in \mathrm{NC}$

Also it can be demonstrated that the formal rules of addition and multiplication hold good for finite cardinal numbers:

*110.51. $\vdash . \mu +_c \nu = \nu +_c \mu .$

*110.56. $\vdash . (\mu +_c \nu) +_c \varpi = \mu +_c (\nu +_c \varpi) .$

*113.27. $\vdash . \mu \times_c \nu = \nu \times_c \mu .$

*113.54. $\vdash . (\mu \times_c \nu) \times_c \varpi = \mu \times_c (\nu \times_c \varpi) .$

*113.43. $\vdash . (\nu +_c \varpi) \times_c \mu = \mu \times_c (\nu +_c \varpi) = (\mu \times_c \nu) +_c (\mu \times_c \varpi) .$

From this it follows that Russell's and Whitehead's formal definitions of addition and multiplication are equivalent to the mathematicians's definitions.

We now have all of the information necessary for the construction of the natural series of natural numbers by beginning with 0 and adding 1 successively to each number. Thus we have:

0+1=1

1+1=2

2+1=3

3+1=4

........etc.

This series, according to Jørgensen, is characterized by the following fifteen propositions, regardless of whether the series is regarded as composed of cardinals or ordinals:[1]

(1). 0 is a natural number.

(2). Every natural number is by another natural number.

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1. These are given verbatim from Jørgensen, op. cit., Vol. 3, pp. 68-69.

(3). Every natural number is identical with itself.

(4). Every natural number is greater than every preceding and less than every succeeding number.

(5). If a system of numbers to which 0 belongs, has the property that, if it contains a number <u>n</u> it also contains <u>n</u>+1, then it contains every natural number.

If a, b, and c are any natural numbers, then:

(6). If a = a' and b = b', then a + b = a' + b.

(7). (x + b) + c = a + (b + c)

(8). a + b = b + a

(9). If a > a', then a + b > a' + b

(10). a + b ≥ a

(11). If a = a' and b = b', then a · b = a' · b'

(12). (a + b) · c = a · c + b · c

(13). (a · b) · c = a · (b · c)

(14). a · b = b · a

(15). If a > a', then a · b > a' · b.

In these propositions we have some expressions and signs which we must define, namely, ≥ (greater than or equal to), > (greater than), "succeeding," "preceding," and "natural number." In <u>Principia</u> we find the following definitions:

$*117.01.\ \mu > \nu . = . (\exists \alpha, \beta) . \mu = N_0c`\alpha . \nu = N_0c`\beta . \exists ! Cl`\alpha \cap Nc`\beta . \sim\exists ! Cl`\beta \cap Nc`\alpha \quad Df.$

$*117.05.\ \mu \geq \nu . = : \mu > \nu . v . \mu, \nu \in N_0C . \mu = sm``\nu \quad Df.$

from which we can deduce the fundamental properties of these relations:

$$*117.281.\ \vdash.\mu>\nu.\equiv.\mu\geq\nu.\sim(\nu\geq\mu).$$

$$*117.4.\ \vdash.\mu\geq\nu.\nu\geq\varpi.\supset.\mu\geq\varpi.$$

$$*117.471.\ \vdash.\mu>\nu.\nu>\varpi.\supset\mu>\varpi.$$

But before we go furhher, we need to explain two symbols used above in*117.01. First $\cap$, which means the logical product or the common part of two classes. A similar symbol $\cup$, means the logical sum of two classes. The former is stated in <u>Principia</u> as follows:

$$*22.02.\ \alpha\cap\beta=\hat{x}(x\varepsilon\alpha.x\varepsilon\beta).\ Df.$$

The latter is:

$$*22.03.\ \alpha\cup\beta=\hat{x}(x\varepsilon\alpha.\vee.x\varepsilon\beta)\ Df$$

The other symbol which needs explaining is Cl, or Cl'α.
Cl is a relation defined as follows:

$$*60.01.\ Cl=\hat{\kappa}\hat{\alpha}\{\kappa=\hat{\beta}(\beta\subset\alpha)\}\ Df.$$

Cl is the relation to a class of the class of all its sub-classes. The sub-classes of a class are all the classes that can be formed from members of the class. The number of sub-classes of a given class is always greater than the number of members of the class. Cl of a given class is an important function of the class. Cl'α is the class of sub-classes of α.

Every natural number is such that $\mu+1>\mu$ since it can be proved that

$$*117.6.\ \vdash:\mu,\nu\varepsilon N_0C.\supset.\mu+_c\nu\geq\mu.\mu+_c\nu\geq\nu.$$

where the equality sign, granted that we are dealing with natural numbers, holds good only if $\nu = 0$. Thus we can place the natural numbers in a series according to magnitude with 0 as the first term and every term a unit greater than the preceding one or a unit less than the succeeding one. But we still have to give the definition of "natural number," or "inductive number" as it is called in the Principia. The definition is given as follows:

*120.01. NC induct $= \hat{\alpha}\{\alpha(+_c D'0)\}$[1] Df.

or in other words the natural or inductive cardinal numbers are the posterity of 0 in regard to the relation n and n + 1.

All of the fifteen propositions given above which characterize the series of natural (cardinal and ordinal) numbers are proved or can easily be proved from the propositions in Principia. In the following list, the first number (like 1, 2, 3, etc.) refers to the proposition of that number in the above list of fifteen propositions, and the next number is the number in Principia.

For (1) we have:

*120.12. $\vdash . \; 0 \varepsilon$ NC induct.

For (2):

*125.12. $\vdash$:. Infin ax. $\equiv$: $\alpha \, \varepsilon$ NC induct. $\supset_\alpha . \exists ! \, \alpha +_c 1$.

(3) follows from the definition of cardinal numbers in conjunction with:

*100.321. $\vdash : \alpha \, sm \, \beta \, . \supset . \, Nc'\alpha = Nc'\beta$.

(4) follows from:

1. D'0 means the field of 0, that is the inductive field.
2. Infin. ax means the axiom of infinity--the assumption that infinity exists.
3. The symbol "sm" means similar.

*120.429. $\vdash :. \nu \varepsilon \text{NC induct} . \supset : \mu > \nu . \equiv . \mu \geq \nu +_c 1.$

For (5) we have:

*120.101. $\vdash :: \alpha \varepsilon \text{NC induct} : \equiv : \xi \varepsilon \mu . \supset_\xi . \xi +_c 1 \varepsilon \mu : 0 \varepsilon \mu : \supset_\mu . \alpha \varepsilon \mu .$

(6) follows from:

*110.15. $\vdash : \gamma \,\text{sm}\, \alpha . \delta \,\text{sm}\, \beta . \supset . \gamma + \delta \,\text{sm}\, \alpha + \beta .$

For (7) we have:

*110.56. $\vdash . (\mu +_c \nu) +_c \varpi = \mu +_c (\nu +_c \varpi) .$

For (8):

*110.51. $\vdash . \mu +_c \nu = \nu +_c \mu .$

(9) follows from:

*117.561. $\vdash : \mu \geq \nu . \varpi \varepsilon N_0C . \supset . \mu +_c \varpi \geq \nu +_c \varpi .$

For (10) we have:

*117.6. $\vdash : \mu , \nu \varepsilon N_0C . \supset . \mu +_c \nu \geq \mu . \mu +_c \nu \geq \nu .$

(11) follows from:

*113.13. $\vdash : \alpha \,\text{sm}\, \gamma . \beta \,\text{sm}\, \delta . \supset . \alpha \downarrow ; {}^{\prime\prime}\beta \,\text{sm}\, \gamma \downarrow ; {}^{\prime\prime}\delta . (\beta \times \alpha) \,\text{sm}\, (\delta \times \gamma) .$

For (12) we have:

*113.43. $\vdash . (\nu +_c \varpi) \times_c \mu = \mu \times_c (\nu +_c \varpi) = (\mu \times_c \nu) +_c (\mu \times_c \varpi) .$

For (13):

*113.54. $\vdash . (\mu \times_c \nu) \times_c \varpi = \mu \times_c (\nu \times_c \varpi) .$

For (14):

*113.27. $\vdash . \mu \times_c \nu = \nu \times_c \mu .$

(15) follows from:

*117.571. $\vdash : \mu \geq \nu . \varpi \varepsilon N_0C . \supset . \mu \times_c \varpi \geq \nu \times_c \varpi .$

We can now conclude with certainty that the whole theory of natural numbers can be, or rather have been , deduced by purely logical methods from the logical calculus given by Russell and Whitehead. From the theory of natural numbers the theory of relative, rational, real and complex numbers can be deduced by various methods.[1] It remains only for us to examine the theory of infinite numbers in order to determine whether or not Russell's proof of his thesis is fundamentally valid.

B. Theory of Infinite Numbers.

Our problem is to determine whether or not ~~in-finite~~ infinite numbers can be defined in purely logical terms. However, this really constitutes two problems, namely, can infinite cardinal numbers be so defined? and can infinite ordinals be so defined? In our discussion of natural numbers, this distinction was not necessary since from the mathematical point of view finite cardinals and finite ordinals follow from the same laws. But this is not true with the infinite cardinals and ordinals.

Let us consider infinite cardinal numbers first. Russell's definition of cardinal numbers as classes of classes which are similar to each other is designed for both finite and infinite cardinal numbers. Therefore, we already have a purely logical definition of infinite cardinal numbers, and it is necessary only for us to determine whether

1. See O. Hölder, Die Arithmetik in strenger Begrundung and Stolz and Gmeiner, Theoretische Arithmetik.

or not the objects defined have the properties attributed to infinite cardinal numbers by mathematicians.

We have to content ourselves with determining whether or not the principle axioms of the theory of infinite cardinal numbers are contained in, or are deducible from, the propositions of the *Principia*, since there is no universally accepted system of such axioms. If we find that these principle axioms are included in or implied by the propositions of *Principia*, we may reasonably conclude that the objects defined by Russell's[1] definition of infinite cardinal numbers do posses the properties attributed by mathematicians to such numbers. The most important propositions concerning the infinite cardinal numbers, which can be proved independently from the infinite ordinal numbers, are the following according to Jørgensen: (1). the theorem of equivalence, (2) the theorem of inequality, (3) the various rules of calculation for infinite cardinal numbers, and (4). the principle propositions dealing with $\aleph$ (aleph).

(1). The theorem of equivalence appears in the *Principia* in the following form:

$$*117.23.\ \vdash :\ Nc`\alpha \geq Nc`\beta \,.\, Nc`\beta \geq Nc`\alpha \,.\equiv.\ Nc`\alpha = Nc`\beta.$$

(2). The theorem of inequality is found likewise in *Principia* as follows:

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1. We say Russell instead of Russell-Whitehead because the same definition as we find in *Principia* was given by Russell in the *Principles* in 1903.

*117.66. $\vdash . Nc`Cl`\alpha > Nc`\alpha$.

*117.661. $\vdash . \mu \varepsilon NoC . \supset . 2^{\mu} > \mu$.

(3). The formal rules of calculation (the commutative and associative principles, etc) noted above in Section A apply also to infinite cardinals since they are proved in Principia on the basis of the general definition of cardinal numbers without distinction between finite and infinite cardinals.

(4). Finally, we come to $\aleph_0$ which is defined as follows:

*123.01. $\aleph_0 = D``\text{Prog}$

where "Prog" means "progression," meaning any series similar to the natural series of numbers. Concerning $\aleph_0$, we have the following propositions proved in the Principia:

*123.36. $\vdash . \aleph_0 \varepsilon NC$

*123.13. $\vdash : \alpha \varepsilon \aleph_0 . \supset . Nc`\alpha = Nc`\alpha +_c 1$,

from which it follows that $\aleph_0$ is not an inductive number (cardinal).

*123.43. $\vdash :. \exists ! \aleph_0 . \supset : \nu \varepsilon NC \text{ induct} . \supset_{\nu} . \aleph_0 > \nu$

that is $\aleph_0$ is greater than any inductive cardinal numbers.

*123.18. $\vdash : \exists ! \aleph_0 (x) . \supset . \text{Infin ax} (x)$.

*123.31. $\vdash : \alpha \varepsilon \aleph_0 . \supset . \alpha \text{ sm } NC \text{ induct}$.

*123.4. $\vdash . \aleph_0 = \aleph_0 +_c 1$.

*123.41. $\vdash : \nu \varepsilon NC \text{ induct} . \supset . \aleph_0 = \aleph_0 +_c \nu$.

*123.411. $\vdash : \nu \varepsilon NC \text{ induct} . \supset . \aleph_0 = \aleph_0 -_c \nu$.

*123.421. $\vdash . \aleph_0 = \aleph_0 +_c \aleph_0 = 2 \times_c \aleph_0$.

$*123.52. \vdash . \aleph_0 = \aleph_0 \times_c \aleph_0 = \aleph_0^2.$
$*124.12. \vdash . \aleph_0 \subset \text{Cls refl}.$
$*124.23. \vdash : \mu \varepsilon \text{NC refl} . \equiv . \mu \geq \aleph_0 .$
$*124.56. \vdash : \aleph_0 \varepsilon \text{NC mult} . \supset . \text{Cls induct} = \text{Cls refl} . \text{N}_0\text{C} - \text{NC induct} = \text{NC refl},$

which means that it makes no difference whether we define the infinite cardinal numbers as reflexive or non-inductive.

So far in our examination of the mathematical properties of infinite cardinal numbers, we have found nothing that is incompatible with the purely logical definition of such numbers given by Russell. However, there still are two important propositions, namely, Cantor's theorem and the theorem of comparison, which we must examine before we can be sure that the theory of infinite cardinal numbers can be deduced from the Russell-Whitehead logical calculus, but, in the Principia, these two theorems are regarded as provable only in conjunction with the theory of infinite ordinal numbers. Therefore, we now pass to the consideration of these.

Our first question is: what is an ordinal number? We did not raise this question in our consideration of the theory of natural numbers because it was not necessary to make a distinction between cardinals and ordinals at that time. In Principia an ordinal number is defined as the ordinal type or the "relation number" of a well-ordered aggregate (or manifold) or series. This is stated symbolically in this manner:

$*251.01.\ NO = Nr``\Omega$ (where NO = ordinal number)

where "Ω" denotes the class of well-ordered series, that is a series in which every existing class has a "minimum" or a first term. This is expressed as follows:

$*250.01.\ Bord = \hat{P}(Cl\ ex`C`P \subset D`min_p)$[1] Df

so that "Ω" is defined:

$*250.02.\ \Omega = Ser \cap Bord$ Df

where "Ser" (series) is defined as an asymmetrical, transitive and connected relation:

$*204.01.\ Ser = Rl`J \cap trans \cap Connex$ Df

We can rightly regard this definition of ordinal numbers as most probably correct if the most important properties of the ordinal numbers can be deduced from it by purely logical means. But to determine this we must distinguish between finite and infinite ordinal numbers; the former being the relation numbers of finite well-ordered series, and the latter being the relation numbers of infinite well-ordered series. These definitions are stated symbolically as follows:

$*262.01.\ NO\ fin = N_0r``\Omega\ fin$ Df

$*262.02.\ NO\ infin = N_0r``\Omega\ infin$ Df

where "Ω fin" and "Ω infin" are defined as follows:

$*261.04.\ \Omega\ fin = \Omega - \Omega\ infin$ Df

$*261.02.\ \Omega\ infin = \Omega \cap \breve{C}``Cls\ refl$ Df

The finite ordinal numbers have precisely the same properties as the finite cardinal numbers. On the basis of

1. "Bord" is an abbreviation for "bene ordinata" or "bien ordonnée," which means class of "well-ordered" series.

$$*262.15. \vdash :. \alpha \in N_0 O . \supset : \alpha \in NO \text{ fin} . \equiv . C``\alpha \in NC \text{ induct}.$$

it can be proved that there is a one-one correspondence between the inductive cardinal numbers and the finite ordinals. Also it is possible to reduce all relations between finite ordinal numbers to relations between corresponding cardinal numbers by means of the following proposition:

$$*262.23. \vdash :. P, Q \in \Omega \text{ fin} . \supset : C`P \text{ sm } C`Q . \equiv . P \text{ smor } Q$$

Since the properties of the finite ordinals are exactly the same as those of the finite cardinals, and since we have already proved the properties of the latter to be properties possessed by the objects defined by the logical definition of cardinal numbers, there is no need for us to prove step by step that these properties can be deduced from the logical definition given of finite ordinals.

But this relation between the ordinals and cardinals does not hold in regard to infinite ordinals. For example, the commutative principle of addition and multiplication does not apply to infinite ordinals, and the distributive principle holds only in the form:

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$$

and not in the form:

$$(\beta + \gamma)\alpha = \beta\alpha + \gamma\alpha$$

The formal principle for addition and multiplication of infinite ordinals apply to ordinal types as a whole and to relation numbers.[1] The sum and product of two relation numbers are defined as follows:

$$*180.02.\ \mu \dot{+} \nu = \hat{R}\{(\exists P, Q).\mu = N_0 r`P.\nu = N_0 r`Q.R \ \mathrm{smor}\ (P+Q)\} \quad \mathrm{Df.}$$

$$*184.01.\ \mu \dot{\times} \nu = \hat{R}\{(\exists P, Q).\mu = N_0 r`P.\nu = N_0 r`Q.R \ \mathrm{smor}\ (P \times Q)\} \quad \mathrm{Df}$$

And it is proved that:

$$*180.42.\ \vdash.\mu \dot{+} \nu \in NR$$

$$*184.15.\ \vdash.\mu \dot{\times} \nu \in NR$$

Also that:

$$*180.56.\ \vdash.(\mu \dot{+} \nu) \dot{+} \varpi = \mu \dot{+} (\nu \dot{+} \varpi)$$

$$*184.31.\ \vdash.(\mu \dot{\times} \nu) \dot{\times} \varpi = \mu \dot{\times} (\nu \dot{\times} \varpi)$$

$$*184.35.\ \vdash.(\nu \dot{+} \varpi) \dot{\times} \mu = (\nu \dot{\times} \mu) \dot{+} (\varpi \dot{\times} \mu)$$

These propositions concerning relation numbers in general apply to infinite ordinals since infinite ordinals are a special kind of relation numbers.

The ordinal type of the natural series of numbers is denoted by ω, and it can be proved that ω is the smallest infinite ordinal. The Principia calls series which are similar to the natural series of numbers "progressions," and therefore ω is defined as follows:

$$*263.01.\ \omega = \hat{P}\{(\exists R).R \in \mathrm{Prog}.P = R_{po}\} \quad \mathrm{Df.}$$

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1. The differences betwwen ordinal numbers and ordinal types and relation numbers are: (1). Ordinal numbers are the relation numbers of well-ordered manifolds (or series), that is classes of all relations similar to a given well-ordered series. (2). Ordinal types are the relation numbers of ordered series; that is classes of all relations similar to a given serial relation. (3). Relation numbers are the classes of all relations similar to a given relation. See Jørgensen, Vol. 3, p. 78, f.n. 28.

It is proved that ω is an ordinal number:

*263.2. $\vdash . \omega \varepsilon NO$

that ω is an infinite ordinal number:

*263.24. $\vdash : \exists ! \omega . \supset . \omega \varepsilon NO\ infin,$

that every ordinal less than ω is finite:

*263.33. $\vdash : \alpha < \omega . \supset . \alpha \varepsilon NO\ fin,$

that ω is the smallest infinite ordinal:

*263.54. $\vdash : \alpha \varepsilon NO\ infin - \iota^{\prime}\omega . \supset . \alpha > \omega$

and that

*263.34. $\vdash . 1 \dot{+} \omega = \omega$

*263.35. $\vdash : \alpha \varepsilon NO\ fin . \supset . \alpha \dot{+} \omega = \omega$

*263.66. $\vdash : \alpha \varepsilon NO\ fin - \iota^{\prime}Or . \supset . \omega \dot{\times} \alpha = \omega$

However, where $\alpha \neq Or$, $\omega \dot{+} \alpha > \omega$ ~~and~~ and this holds good for all ordinal numbers. The following propositions of the Principia express this:

*255.32. $\vdash :. \gamma, \varpi \varepsilon N_0 O . \supset : \gamma \dot{+} \varpi > \gamma . \equiv . \varpi \neq Or$

*255.571. $\vdash : \alpha, \beta \varepsilon N_0 O - \iota^{\prime}Or . \supset . \beta \leq \alpha \dot{\times} \beta$

*255.321. $\vdash :. \gamma \varepsilon N_0 O . \supset : \gamma \neq Or . \equiv . \gamma \dot{+} 1 > \gamma$

The unlimited series of infinite ordinals can be constructed without difficulty in principle, as follows:[1]

$\omega, \omega+1, \omega+2 \ldots \omega \cdot 2, \omega \cdot 2+1, \ldots \omega \cdot 3, \ldots \omega \cdot 4 \ldots \omega \cdot m+n,$
$\omega^2, \omega^2+1, \ldots \omega^2+\omega, \ldots \omega^2+\omega \cdot 2, \ldots \omega^2 \cdot 2, \ldots \omega^2 \cdot m+\omega \cdot n+p, \ldots$
$\omega^3, \omega^3+1, \ldots \omega^n, \ldots (\omega^n \cdot m_n+\omega^{n-1} \cdot m_{n-1}+\omega^{n-2} \cdot m_{n-2}+ \ldots$
$\omega \cdot m_1 + m_0), \ldots \omega^\omega, \omega^{\omega+1}, \omega^\omega+\omega^\omega, \ldots \omega^\omega \cdot n, \ldots \omega^{\omega+1},$
$\omega^{\omega+1}+1, \omega^{\omega \cdot n} \ldots \omega^{\omega^2} \ldots \omega^{\omega^n}, \omega^{\omega^{\omega \cdots}} (= \varepsilon), \varepsilon+1, \ldots$ etc.

1. See Fraenkel, Einleitung, etc., p. 132, and Jørgensen, Vol. 3., p. 79.

There is a corresponding cardinal number for each of these ordinals, namely, the cardinal number of the corresponding well-ordered series. Thus the cardinal number $\aleph_0$ corresponds with ω in that

$$*263.101. \vdash . \aleph_0 = D``\omega = C``\omega$$

and the existence of ω and $\aleph_0$, each of which assumes the axiom of infinity, is proved in the Principia to be equivalent within every logical type:

$$*263.131. \vdash : \exists ! (\aleph_0)_\alpha . \equiv . \exists ! \omega \cap T_0{}^{\prime\prime}\alpha$$

But there are infinite numbers of infinite ordinals answering to $\aleph_0$ since a manifold of the power $\aleph_0$ can be ordered in an infinite number of ways. The manifold of all this infinite number of infinite ordinals corresponding to $\aleph_0$ is known as the "number class" corresponding to $\aleph_0$ and ω, the smallest number in this number class, is called its initial number. All the terms in this series of infinite ordinals are members of this number class, but together they have a power that is greater than $\aleph_0$ and this greater power is $\aleph_1$. Again the elements of this class can be arranged in an infinite number of ways, with an ordinal number for each way, and the manifold of these makes up a new number class. The initial number of this new class is ω_1. This process of constructing successive powers and number classes can be carried on ad infinitum.

The series of initial numbers and alephs is introduced in Principia by the following definitions:

*265.01. $\omega_1 = \hat{P}\{\overrightarrow{\text{less}}{}^{\backprime}P = (\aleph_0)_r \cup \Omega \text{ fin}\}$ Df.

*265.02. $\aleph_1 = C^{\backprime\backprime}\omega_1$ Df.

*265.03. $\omega_2 = \hat{P}\{\overrightarrow{\text{less}}{}^{\backprime}P = (\aleph_1)_r \cup (\aleph_0)_r \cup \Lambda \Omega \text{ fin}\}$ Df.

*265.04. $\aleph_2 = C^{\backprime\backprime}\omega_2$

etc.

In other words, ω_1 "is the class of relations (series) which are so constituted that every series which is similar to a part of and only a part of the first mentioned series is a well-ordered finite series or well-ordered series having $\aleph_0$ terms" and $\aleph_1$ "is the class of those classes which can be arranged in series whose ordinal number is ω_2."[1]

These numbers can be proved to be ordinals and cardinals respectively, since the following propositions are proved in Principia:

*265.12. $\vdash . \omega_E \, \varepsilon \, NO$

*265.33. $\vdash . \aleph_1 \, \varepsilon \, NC$

But for the existence of these numbers, we have to accept it on the basis of the axiom of infinity, since their existence cannot be proved.

Concerning the axiom of infinity, it is stated symbolically in Principia as follows:

*120.03. Infin ax. = : $\alpha \, \varepsilon$ NC induct. $\supset . \exists ! \alpha$ Df.

which says that, if α is an inductive cardinal number, there is at least one class of the type in question which has α

1. Jørgensen, op. cit., Vol. 3, p. 80.

terms. This is an existence theorem, and it is applicable to ordinals also. Concerning it the authors say: "This assumption...will be adduced as a hypothesis whenever it is relevant. It seems plain that there is nothing in logic to necessitate its truth or falsehood, and that it can only be legitimately believed or disbelieved on empirical grounds."[1]

So far we have considered only ordinal numbers, that is the ordinal types of well-ordered manifolds, but the theory of manifolds has other types also, the most important of which are those designated by Cantor by the symbols η and θ, the former being the ordinal type of the manifold of rational numbers and the latter the ordinal type of the linear continuum. We must determine whether or not Russell and Whitehead's definitions of these types are correct and this can only be done by determining whether or not the characteristic properties of the types can be deduced from the given definitions.

The ordinal type η is defined in <u>Principia</u> as follows:

$$*273.01.\ \eta = Ser \cap comp \cap \overleftarrow{C}``\aleph_0 \cap \hat{P}(D`P = G`P)\ Df.$$

which means that η is a compact series without a beginning or end and contains $\aleph_0$ terms. This corresponds exactly with Cantor's characterization of this type.

The following propositions can then be proved about η:

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1. <u>Principia</u>, Vol. 2, p. 189.

*273.4. $\vdash : P, Q \varepsilon \eta . \supset . P \text{ smor } Q$

*273.41. $\vdash : P \varepsilon \eta . P \text{ smor } Q . \supset . Q \varepsilon \eta .$

*273.43. $\vdash . \eta \varepsilon NR$

(No number): $\vdash : \alpha \varepsilon NR \cap Cl'Ser . C''\alpha = \aleph_0 . \supset . \alpha \dot{\times} \eta = \eta$

*274.43. $\vdash . \aleph_0 = C''\eta$

*274.42. $\vdash : \alpha \varepsilon \aleph_0 . \supset . \exists ! \eta \cap \overleftarrow{C}'\alpha$

*304.33. $\vdash$: Infin ax. $\supset . H \varepsilon \eta$

Thus, assuming the validity of the axiom of infinity, we find that the series of rational numbers has the ordinal type η .

The ordinal type θ , the ordinal type of linear continuity, is characterized, according to Cantor, by the fact that it is perfect and contains a manifold with the cardinal number $\aleph_0$ so that elements of this manifold lie between every two elements of θ . A perfect manifold is one that is closed and dense in itself. Such a manifold is said by Russell and Whitehead to have "Dedekind Continuity."

We must consider briefly what the phrase "Dedekind Continuity" means. Dedekind[1] gives a definition of the continuity of a straight line as follows: "If all the points of a line can be divided into two classes such that every point of one class is to the left of every point of the other class, then there exists one and only one point which brings about this division of all points into two classes, this section of the line into two parts." This definition,

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1. Dedekind, Stetigkeit und irrationale Zahlen (Brunswick, 1892, 2nd. edition), p. 11. Quoted by Russell in Principles, p. 279.

which is known as the Dedekind axiom of continuity, is stated rather loosely. Russell[1] states what he thinks that Dedekind meant by the axiom in the following words: "A series, we may say, is continuous in Dedekind's sense when, and only when, if all the terms of the series, without exception, be divided into two classes, such that the whole of the first class precedes the whole of the second, then, however the division be effected, either the first class has a last term, or the second class has a first term, but never both. This term, which comes at one end of one of the two classes, may then be used, in Dedekind's manner, to define the section."

In addition to the concept of "Dedekind Continuity," a definition of "median" class is required before Θ can be defined. Russell and Whitehead say:[2] "We shall call a class α a 'median' class in P [where P is a series] if $\alpha \subset C^{\prime}P$ and there is a member of α between any two terms of which one has the relation P to the other." This definition is stated symbolically as follows:

$$\text{med} = \hat{\alpha}\hat{P}(\alpha \subset C^{\prime}P \,.\, P \subset P \upharpoonright \alpha \backslash P) \quad \text{Df.}$$

By virtue of the Dedekind axiom of continuity and the definition of a "median class," the authors define Θ as follows:

$$*275.01.\ \Theta = \text{Ser} \cap \text{Ded} \cap \breve{\text{med}}\,``\aleph_0$$

where it is possible to prove that

$$*275.3.\ \vdash : P, Q \in \Theta \,.\, \supset .\ P \,\text{smor}\, Q$$

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1. Principles, pp. 279-280.
2. Principia, Vol. 3, p. 186.

$*275.31.\ \vdash : P \varepsilon \Theta \,.\, P \text{ smor } Q \,.\supset .\, Q \varepsilon \Theta$

$*276.33.\ \vdash .\, Q \varepsilon NR$

$*276.43.\ \vdash .\, C``\Theta = 2^{\aleph_0}$

$*310.15.\ \vdash : \text{infin ax} \,.\supset .\, \bar{\Theta}' \leftrightarrow C`H,\ \bar{\Theta}'_n \leftrightarrow C`H_n,\ C`H_n \leftrightarrow \bar{\Theta}_g \leftrightarrow C`H \varepsilon \Theta.$

From this it follows that the ordinal type of $\bar{\Theta}$ is equal to Θ exclusive of 0 and ∞, which corresponds with the fact that $\bar{\Theta}$ is defined as the series of positive real numbers excluding 0 and ∞.

It may now be concluded that the theory of ordinal numbers and other ordinal types can be deduced from the primitive ideas and primitive propositions plus the axiom of infinity by purely logical means.

In addition to the axiom of infinity, Russell and Whitehead employ another axiom, which is not among the primitives of the logical calculus and is not proved, in their deduction of pure mathematics from the primitive ideas and propositions by logical means. This second axiom, which is equivalent to what is known to mathematicians as Zermelo's axiom, is called the Multiplicative axiom in Principia. This axiom is defined as follows:

$*88.03.$ Mult ax. = : $\kappa \varepsilon$ Cls ex^{2}encl. $\supset$:
$(\exists \mu) : \alpha \varepsilon \kappa \,.\supset_\alpha .\, \mu \cap \alpha \varepsilon 1$ Df.

Concerning this, the authors say:[1] "This axiom is equivalent to the assumption that an arithmetical product cannot be zero unless one of its factors is zero, and is regarded by some mathematicians as a self-evident truth. This can be proved when the number of factors is finite...but not when the number of factors is infinite. We have not assumed its truth in the general case when it cannot be proved, but have included it in the hypothesis of all propositions which depend upon it."

We have now seen that Russell and Whitehead's deduction of the theory of natural numbers, from which the theories of relative, rational, real and complex numbers can be easily deduced, and the theory of infinite numbers, both cardinal and ordinal, from the primitive ideas and the primitive propositions of the logical calculus (plus the axiom of infinity and the multiplicative axiom) by purely logical means is valid.

Geometry, as we have pointed out previously, has not been subjected to detailed treatment, but from the work done on the subject in the Principles, we may conclude that in all probability geometry follows from the logical calculus in much the same way as the theory of natural numbers and the theory of infinite numbers.

Although the axiom of infinity and the multiplicative axiom have to be assumed, geometry has not yet been deduced step by step from the logical calculus, and the axiom of reducibility is not satisfactory, we may safely conclude that in all probability these difficulties can be cleared up and thus that Russell's proof is fundamentally valid, and, consequently, that his thesis is established.

Chapter VIII

The Philosophical Importance of Mathematical Logic

By the way of summary, we have seen that the thesis which runs through all of Russell's logico-mathematical works is the contention that logic and pure mathematics are one in the sense that they constitute a continuous whole with logic coming first and mathematics second, but that there is no particular point of which it can be said that logic ends and mathematics begins. We traced the development of this thesis from its explicit origin in Leibniz down through the development of logicians like Boole, Schroeder, Peirce, Frege, Peano to Russell and Whitehead and such mathematicians as Weierstrass, Dedekind, Cantor, and Russell. The logicians were deducing logic to an arithmetical form by the development of symbolism and the mathematicians were generalizing or logicalizing mathematics. These two processes continued without full apprehension by those who were carrying out the processes until the two lines of development merged in Russell who was both a master mathematician and logician. Many of the results of the thesis were worked out before the thesis itself was explicit to say nothing of its proof; for example, Weierstrass' abolition of the infinitesimal, Dedekind's and Cantor's definitions of the mathematical infinite and continuity. But these came into more certainty and fuller meaning with the work of Russell and Whitehead. We saw how

Russell defined pure mathematics as the class of propositions which are expressed exclusively in terms of variables and logical constants, that is to say as the class of purely formal propositions. We have traced the development of Russell's logical calculus from which he deduces mathematics by purely logical means. Although the number of primitive ideas and primitive propositions vary at different times in his system, they remain a very few at all times. We discussed the theory of types and how they resolve the logical paradoxes resulting from the vicious-circle fallacy. We outlined Russel's proof of his thesis and examined his and Whitehead's demonstration of it. We concluded that Russell's proof of his thesis is valid and consequently that his thesis is well established.

We must now consider the philosophical importance of mathematical logic, which, although it had it beginning with Leibniz and has been contributed to by many, has received its fullest development and proof by Russell in collaboration with Whitehead. Concerning it, Russell says:[1]

> It has, in my opinion, introduced the same kind of advance into philosophy as Galileo introduced into physics, making it possible at last to see what kinds of problems may be capable of solution, and what kinds must be abandoned as beyond human powers. And where a solution appears possible, the new logic provides a method which enables us to obtain results that do not merely

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1. Russell, Our Knowledge of the External World (Chicago: The Open Court Publishing Co., 1914), P.59.

embody personal idiosyncrasies, but must command the assent of all who are competent to form an opinion.

This is rather a generous appraisal of the significance of mathematical logic, but let us consider the several contributions that it has made to philosophy.

1. The first significance of mathematical logic is that it gives us a new logic. Traditionally, logic was concerned with classes, propositions and syllogisms. For many it became a means of refuting common sense. Aristotle had spoken and no one dared to question the judgments of the great master. But finally a new logic was born out of the examination of mathematics. This was begun by Boole in 1854 and the most important contributors to the development of the new logic have been mathematicians. The fundamentals of this new logic are propositions, propositional functions, classes, and relations. Propositional functions and relations are the two most important introductions into the new logic and Russell is largely responsible for both of them. We have discussed this matter elsewhere, but at this point it is necessary for us to consider briefly Russell's treatment of propositions and relations.

Traditionally, logicians divided all propositions into subjects and predicates. This left out the verb entirely. In the proposition "Socrates is mortal," the subject is "Socrates" and the predicate is "mortal". Thus the proposition is analyzed into a subject and a quality of the subject, but nothing is done with the verb. Russell chose to analyze such a proposition into a subject and an assertion, with the assertion being "is mortal." In this way all of the elements of the proposition are accounted for in the analysis.

According to traditional logic a proposition had to be true or false. The law of the excluded middle ruled out any neutral ground. Yet there appeared to be statements of the propositional form which seemed to be neither true nor false. Russell called such statements propositional functions, that is, statements containing a variable which become propositions upon the insertion of a constant in place of the variable. Such statements may be said to be neither true nor false and as they become propositions they are said to be true at one time and false at another according to the value given to the variable. This made propositional forms much more flexible.

Traditional logic was unable to admit the reality of relations since it held that all propositions are of the subject-predicate form. This made it necessary to reduce relations to properties of the terms related. Thus there could be no calculus of relations since they were included in the calculus of propositions. Peano's system was incomplete at this point. Schroeder and Peirce developed the logic of relations to a great extent and Russell worked it into his system. Traditional logic reduced everything to subjects and qualities of the subjects. The new logic reduces everything to subjects, qualities, and relations. As we shall observe later, this results in significant philosophical consequences.

In addition to these fundamentals of the new logic, a rather complete and detailed logical symbolism has been developed. In a true sense this is a science and not part of philosophy. It began as philosophy, but the preciseness of its symbols and laws promoted it from the area of the controversial to the area of certainty. Yet this science of symbolic demonstration is a product of philosophy and is of great philosophical importance. The subject matter with which it is concerned was part of the subject matter of philosophy prior to the development of mathematical logic. Thus, the first part of the new logic is philosophical and the second part is mathematical.

2. Mathematical logic has provided a sound philosophical basis for mathematics. Prior to the development of this new logic mathematics was regarded as largely empirical. It was defined as the science of quantity and measurement. Number was regarded as self-evident and indefinable. Geometry was thought to deal with physical space composed of points. Each branch of mathematics had its own set of indefinable ideas and indemonstrable propositions. But mathematical logic has made it possible to give a purely formal definition of mathematics, to make pure mathematics independent of quantity and measurement, to define in purely logical

terms all of the fundamental mathematical concepts, and to deduce by logical means all of mathematics from one set of very few indefinable ideas and indemonstrable propositions. Thus mathematics has been given the logical calculus as its foundation and has been unified in that all branches of mathematics are deduced from the one set of primitive ideas and primitive propositions.

3. Mathematical logic has given philosophy a scientific method. Much of the philosophy of the past has been based on either the intuitive, a literary, or an inadequate logical method, if any method at all. Even where an inadequate logical method has been employed, it has, for the most part, been used only to establish logical bases for conclusions reached by the intuitive vision. Philosophy has not had an adequate method of approach to knowledge. Physics until the time of Galileo also lacked an adequate method, and until that time physics was unprogressive, vague and superstitious. But Galileo gave physics a scientific method. He closely observed physical facts and subjected them to mathematical manipulation. With mathematics as its method of research, physics has advanced miraculously. Advances in all branches of science have to wait for adequate advances in mathematics. The new logic proposes to be the method of research in philosophy as mathematics is the method in science.

The way in which the scientific method, based on the new logic, works in philosophy may be briefly summarized.

First of all a mental discipline must be developed. This consists of two processes, namely, doubting the familiar and imagining the unfamiliar. The first process is the Cartesian method. We must not be lead into believing the traditional and the obvious without critical examination. This process of doubting the familiar consists of never taking anything at its face value. The matter of imagining the unfamiliar is to provide as many logically possible hypotheses as possible. When there are a number of possible hypotheses to be examined, it is more likely that one of them is correct. If only one hypothesis appears to be possible, there is a tendency to attempt to make it appear logical. In doing this, logic is used as a defensive weapon rather than as an offensive one. The lack of fertility in imagining abstract hypotheses has been one of the great faults in philosophy heretofore. The traditional logical apparatus was so meager that most of the hypotheses philosophers could imagine were found inconsistent with the facts.

After our problem has been selected and the necessary mental discipline acquired, the method to be followed has been fairly well established. The big problems are found upon examination to be complex and dependent upon a number of component problems. These are more abstract than those of which they are components. In most cases it will be found that all of our initial data, all the facts which we thought that we knew at the beginning of our inquiry, are vague, confused, and complex. It is necessary to create an apparatus of precise conceptions which are abstract and simple before the data can be properly analyzed into the kind of premisses we want. In this process the source of the difficulty is traced back further and further with each stage becoming more abstract and more difficult to apprehend. A number of these purely abstract questions usually underlie any one of the big obvious philosophical problems. Often we reach a stage beyond which it is impossible to go by our method. Then only the philosophical vision of a genius can come to the rescue. What is needed at this point is some new effort of logical imagination, some glimpse of a possibility never conceived before. Failure to think of the right possibility results in insoluble difficulties and varying degrees of bewilderment and despair. But the right possibility usually quickly establishes itself by absorbing what had appeared to be contradictory facts. From this point on the work of the philosopher on the problem ceases to be analytic and becomes synthetic.

This method in philosophy may not be as revolutionary as the method of Galileo proved to be to physics, but it has produced results already, for example, the solution of the problems of number, infinity, continuity, space, time, and motion. What is more, it holds great promises for the future. However, the immediate effect of such a method is to drastically reduce what had heretofore been regarded as known. It cleans house and begins at the bottom. Traditionally philosophy has liked to work on the big problems, but this method makes it concerned with what might appear to be intrinsically trivial, for it knows that knowledge is valuable and may lead to other knowledge regardless of how insignificant it may appear. The procedure is to accumulate a storehouse of well established knowledge regardless of whether it appears to be usable or not. The time may come when it may contribute to the solution of a big problem. This procedure tends to rob philosophy of much of the glamor that it has enjoyed through the centuries, but it makes it more fruitful and more practical.

4. Mathematical logic has made possible the solution of the problems of the infinitesimal, the infinite, and continuity. We have already discussed these in another part of our study, but we shall consider them at this point also.

The infinitesimal was regarded as the infinitely little. This was considered necessary for continuity. Yet it created great difficulties. It was clear the infinitesimal was not zero and the sum of a number of infinitesimals seemed to be finite. Thus Weierstrass abolished the idea entirely and accomplished all of the desired results without it.

In the article "Recent Work on the Principles of Mathematics" in the <u>International Monthly</u>, Russell gives three rather odd sounding consequences of the banishment of of the infinitesimal, which are very interesting. (1) The first of these is the statement that "there is no such thing as the next moment." That which forms the continuity between two moments has been considered as an infinitesimal element of time, but, with the banishment of the infinitesimal, we have to say that between any two moments there is a finite element of time. A finite element is divisible. So between any two moments of time there is a moment in between. Consequently there can be no such thing as the next moment. From this we must conclude that there is an infinite number of finite moments between any two moments. The philosophy of the infinite comes in and solves what at first appears to be a hopeless situation. (2) In the second place, there is no such thing as the next point in space. "Points" have been thought to be infinitesimal lengths. Yet if any piece of matter be halved, and the remainder halved, etc. points would never be reached. Regardless of how small the pieces became, they would always be finite in size. Yet there are points, but they cannot be reached by successive divisions. Here again it is the philosophy of the infinite that shows us how points are possible and why points are not infinitesimal lengths. (3) The third paradoxical result of the abandoning of the infinitesimal is the fact that when a thing changes it is not in a state of change and when a thing moves it is not in a state of motion. All that can be said of a body in motion is that it is in one place at one time and at another place at another time. We cannot say that at the next instant it will be at the next place. At any instant the body is in some one place. Consequently it is not in a state of motion at any moment.

The problem of the infinite has been a stumbling block to mathematicians, philosophers, and theologians through the centuries. No one until the time of Dedekind and Cantor were able to resolve the problems involved. They discovered that the contradictions involved in the idea of the infinite resulted from the common sense axiom that a part cannot have as many terms as the whole of which it is a part. This axiom appears to be perfectly true and it can be proved to be true as far as finite numbers are concerned. But Cantor concluded that it does not hold for infinite collections. So he postulated the paradoxical sounding definition of an infinite collection as one which can be so divided that a part of it has as many terms as the whole collection of which it is a part. This definition sounds like nonsense, but it solves all of the contradictions involved in the idea of the infinite and it bears no evil fruit. So why follow common sense? It seems that among infinites a part can be equal to the whole of which it is a part. For example, there are just as many even finite numbers as there are finite numbers altogether, since every number can be multiplied by two and every such product is an even number. This could not have been accepted on the basis of Aristotelian logic.

A very strange oddity results from this definition of infinity, but it is only an oddity and not a contradiction. It states that if Achilles and the tortoise travelled forever, the tortoise would travel just as far as Achilles. Russell calls this "the paradox of Tristram Shandy".[1] The name "Tristram Shandy" comes from a man by that name who spent two years chronicling the first two days of his life. He then gave up lamenting that at that rate material would accumulate faster than he could record it and as the years passed he would be further from completing his job than he was at the beginning. Russell maintains that, if Tristram had lived forever, and had not wearied of his task, then, even if all the other years of his life had continued to be as eventful as the first two days, no part of his biography would have remained unwritten. In recording two days in two years, he recorded at the ratio of one day's events in one Year's time. Thus the hundredth day of his life would be recorded on the hundredth year of his life, the thousandth day in the thousandth year, etc. <u>ad infinitum</u>. If he lived forever, then no day of his life would remain unrecorded. This paradoxical but perfectly true proposition depends upon the fact that the number of days in all time is no greater than the number of years.

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1. See <u>Int. Monthly</u>, vol. 4, 1901, P.96 and <u>Principles</u>, 358.

The problem of continuity is solved with the solving of the problems of the infinitesimal and infinity. Zeno, as we shall show in a moment, demonstrated that spaces and times could not consist of a finite number of points and instants, respectively, since any space or time can be halved. Thus between any two points of space or instants of time there is a third point or instant. Philosophers believed the infinite number to be self-contradictory. Thus space and time could neither be composed of a finite number of points and instants nor an infinite number. This has been one of the great problems in the history of philosophy. But the solution of the problems of the infinitesimal and infinity has made it possible to believe that space and time consist of an infinite number of finite points and instants.

However, the problem of continuity, as mathematical logic has proved, is not a problem concerned with space and time and their component parts. It is concerned with ordered series. It is only incidental that points and instants, whether they be regarded as actually spatial and temporal or only logical constructions composed of events, happen to form ordered series.

Mathematicians distinguish between different degrees of continuity, but for philosophical purposes all that is important in continuity is introduced in the lowest degree of continuity. Thus continuity is defined as a compact series. Such a series is one in which there is an element between any two elements, or we might as well say that there is an infinite number of elements between any two given elements. These elements are all finite. There is no such thing as a next element in such a series. A simple example of a compact series is the series of fractions in order of magnitude. No two fractions are consecutive. There are always other fractions, an infinite number of other fractions, between any given two.

This definition of continuity is made possible by the abolition of the infinitesimal and the definition of infinity. The elements in the compact series have to be finite instead of being infinitely little or infinitesimal. Also the number of elements between any two elements of an infinite compact series must be as great as the number of elements in the whole series.

This solution of the problem of continuity is not logically possible on the basis of the subject-predicate logic because it denies the reality of relations. The best proof that relations are real and not merely properties of the objects related is asymmetrical relations, which are relations of the type that is such that if a relation holds between A and B, it never holds between B and A. Symmetrical, non-symmetrical, transitive, intransitive, and non-transitive relations can possibly be explained as properties of the terms they relate, but asymmetrical relations, which are such as "before," "after," "greater," "above," "to the right of," etc., cannot possibly be explained as properties of the terms related. They have meaning independently of the terms they relate. Thus they seem to be real in themselves. If they are, then we may conclude that the others are also.

If asymmetrical relations cannot be properties of the terms related, and are not real in themselves, then we must deny asymmetrical relations. If we do this, we cannot have series, since asymmetrical relations give rise to all series. If we cannot have compact series, we cannot have our solution of continuity. But according to mathematical logic, relations are real and thus we have compact series and consequently the problem of continuity is solved.

5. The solutions of the problems of the infinitesimal, infinity, and continuity have made possible a solution for Zeno's paradoxes. Zeno, the Eleatic, in order to prove the difficulties involved in the idea of motion, produced four argumentssto prove that motion is impossible. Each of the four involvesthe same difficulties. Therefore, we will consider only the two most widely known, namely, the paradoxes of Achilles' race with the tortoise and the flying arrow.

Concerning the first, Zeno argued: let Achilles and the tortoise start on a road at the same time, the tortoise being allowed a handicap, Achilles will never reach the tortoise although he may travel twice, ten, or a hundred times as fast, for at every moment the tortoise is in some place and Achilles is in some place, and neither is ever twice in the same place during the race. Thus the tortoise goes to just as many places during the time of the race as Achilles does. But if Achilles caught up with the tortoise he would have been in more places during the race than the tortoise. Here Zeno appealed to the old common sense principle and said that the distance travelled by the tortoise, if Achilles should catbh up with him, which would be only a part of the distance travelled by Achilles, would necessarily have fewer places (or points) in it than the distance of which it is a part, namely, the distance travelled by Achilles. But it was obvious that the number of points they had been was the same since they had been travelling the same number of instants of time and at each instant each of them had been at one and only one point. Thus it was said that it was logically impossible for Achilles to catch up with the tortoise. With the new definition of infinity there is no problem at all. The part travelled by the tortoise had just as many points in it as the greater distance travelled by Achilles. Thus in the same number of moments and being in the same number of points, Achilles could catch up with him since the number of terms of the part can be equal to the number of terms of the whole of which it is a part.

The paradox of the arrow's being at rest in flight is paraphrased by Burnet, in Early Greek Philosophy, as follows: "The arrow in flight is at rest. For, if everything is at rest when it occupies a space equal to itself, and what is in flight at any given moment always occupies a space equal to itself, it cannot move."

The difficulty here is the assumption that a finite part of time consists of a finite series of successive instants. Throughout an instant it is said that a moving body is where it is. At the next instant it is somewhere else, but it got there in some miraculus way outside of time, or that is not at any particular instant. Thus it is never moving but it occupies different positions. The solution of the problem lies in the theory of the continuous series of instants in which there is no next instant just as there is no next point in the continuous compact series of points in space. This theory is made possible by the abolition of the infinitesimal. All that can be said of a body in motion is that it is in one place at one instant and at another place at another instant. We cannot say that at the next instant it will be at the next place or point. At any given instant the body is in some one place.

This seems to give a rather discontinuous picture of motion, but in reality it is continuous. Imagine a tiny speck of light moving along a scale. Any two positions occupied by the speck at any two instants have other intermediate positions occupied at intermediate instants. However close together we take the two positions the speck will not jump suddenly from the one to the other, but rather it will pass through an infinite number of other positions on the way. Every distance, regardless of how small it may be, is traversed by passing through all the infinite series of positions between the two points marking off the distance. The is true of time. There is a one-one correspondence between the points traversed in any given distance and the instants in the time required. Thus motion is possible in a space and time composed of an infinite number of finite points and instants,respectively. This is made possible by the abolition of the infinitesimal, and the theories of infinity and continuity.

6. The new logic has made it logically possible, by the admission of the reality of relations, to construct a philosophy of matter, space, time, and motion in accord with the advances of the new physics. We saw in the preceding section how the new logic made the old physical explanations of space, time, and motion logically possible. But it is now impossible to reconcile the ideas involved in the theory that solid bodies move in a metaphysical space and time with the new physics. Let us outline briefly Russell's philosophy of these concepts, which is made possible by his belief in the reality of relations.

In the first place, he says, following Whitehead, that particles of matter, points, of space, and instants of time are only logical constructions. The realities which they represent in physics are "events." and relations between events. In physics, an "event" is anything which, according to the old notions, has a date and a place. According to the new physics, an event is defined as that which occupies a small finite amount of space-time. An explosion, a flash of lightning, or any occurrence would be an event. If an event has parts, then its parts are events also, but it does not necessarily follow from the fact that events are finite that they have parts. In every event that has parts, there is a minimal event which has no parts, and this event occupies a finite region in space-time. It may overlap with each of ~~the~~ two others, although the first of these others wholly precedes the second. For example, you may hear a long note on the violin while you hear two short ones on the piano. This overlapping makes continuity possible.

A string of events with certain correlations between them make up what is known as the history of one body, or the course of one light wave, etc. The unity of a body is the unity of history. It is like the unity of a tune which takes time to play, and does not exist whole at any one moment. What exists at any one moment is only an event. However, this event may be a collection of a finite number of events. In this sense, events replace what was known, according to the hld notions, as particles of matter.

Events in the physical world have relations to one another which are known as spatial and temporal. These relations have given rise to the notions of space and time. These relations have a relation of order so that we can say that one event is nearer to a second than to a third. In this way we arrive at what is called the "neighborhood" of an event, which consists of the events very near the given event. When relations exist between two events, the nearer the events are to each other, the more nearly they have these relations.

Two "neighboring" events have a relation known as an "interval" which is measurable quantitatively. Sometimes it is analogous to distance in space and sometimes to lapse of time. The former relation is called space-like and the latter is known as time-like. An interval between two events is space-like when one body could be at only one of the two events. It is time-like when one body could be present at both events.

Four numbers are required to fix the position of an event in space-time. They are the numbers corresponding, according to the old notions, to the time and the three dimensions of space. These four are known as the co-ordinates of an event, and they give us what is known as four dimensional space-time. For example, suppose an airplne has an accident in mid-air. To fix its position, you need to establish its latitude, longitude, altitude above sea-level, and Eastern War Time (or any other time).

There are no direct relations between distant events, such as distace in time or space. Bodies take the course of least resistance, according to the nature of space-time in the particular region in which they are. This is called a geodesic.

Motion has been thought to involve the presence of one body in different positions at different times. As we have said before, a body is a series of events connected by certain discoverable laws so that it has enough unity to deserve a single name. We call it by that name and imagine that the string of events concerned is a single thing. If the events are not all in the same place, we say that the thing has moved. But Russell says that "what we call motion of matter really means that the centre of such a set of events at one time does not have the same spatial relations to other events as the connected centre at another time has to the connected other events. It does not mean that there is a definite entity, a piece of matter, which is now in one place and now in another."[1]

Thus according to Russell matter is a series of events connected by certain causal laws, space-time is relations among events, and motion is a mere change of relations. There is no such thing as cosmic space or time. There is nothing but events and relations among events. But again the relations are events also. Therefore, everything is composed of events.

Such a philosophy as Russell has put forth could not have been possible on the basis of the old logic.

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1. Russell, Philosophy (New York: W.W. Norton and Company, Inc., 1927), p. 163.

7. The new logic, by means of the theory of types, has solved the paradoxes resulting from the vicious-circle fallacy. These paradoxes result from propositions, which refer to "all of a kind" being applied to themselves. They result from self-reference. For example, the proposition: "I am lying." If the statement is true, it must be false, and if it is false, it must be true. All such paradoxes result from the proposition itself being inserted as a value of the variable in the proposition's propositional function. The theory of types resolves this problem by setting up ranges of significance or types of order. A proposition is always in a type higher than its own range of significance. Therefore, it cannot refer to itself and have meaning. But the truth-value of the proposition in question is determined by a proposition of the next type in the hierarchy. This theory of types is part of the logical calculus of Russell's logic.

8. Epistemologically, the new logic refutes both strict empiricism and idealism. Mathematical logic requires knowledge that is not based on sense-data, in fact most, if not all, of the primitives of the logical calculus belong in this category. Let us cite only one case, namely, the principle of induction, which is a form of deduction according to Russell. If we say that the extention of a given case to the general, or of the particular to the universal, is effected by the means of induction, we must admit that induction itself is not proved by experience. It is

evident that the principle is general and that it cannot, without a vicious circle, be demonstrated by induction. Thus it is what we call primitive knowledge. All inductive knowledge needs logical principles which are a priori and universal.

Knowledge consists of two kinds, namely, knowledge of particular facts, which alone enables us to affirm existence, and knowledge of logical truths, which alone enables us to reason about data. In daily life and science we find these two intermixed. Before there can be reasoning, there must be self-evident logical truths, that is truths which are known without demonstration. These truths are the premises of pure mathematics and the deductive elements in every demonstration on whatever subject.

This refutes the theory of knowledge held by strict empiricists, but it does not imply that idealism is right. If general truths expressed only psychological facts, we could not know that they would be true for all people or whether they would be constant from one moment to another. What is more, we could never use general truths legitimately to deduce one fact from another since they would not connect facts at all but only our ideas about facts.

Logic and mathematics force us to admit a kind of scholastic realism in which we admit that there is a world of universals which subsits, and a world of particulars which exists. Some of the universals are known a priori as primitive knowledge and the others are deduced from them by the a priori rules of deduction. Therefore, all of the universals known to us are independent of knowledge by experience of the actual world. The world of particulars are known by experience. It is this aspect of knowledge that enables us to affirm existence. The world of universals enables us to reason.

9. Mathematical logic has harmonized logic and sense-data. This has been possible by the admission of the reality of relations and the solutions of the various problems made possible by the new scientific method given to philosophy. The old logic often contradicted sense-data. For example, we have seen how the old logic caused people to declare that either space and time are mere illusions or that they are not composed of points and instants. It is true that on the basis of the new physics, many philosophers, including Russell himself, admit that there is no metaphysical space or time, points and instants, except as logical constructions composed of events and their relations. However, the new logic makes both theories possible and both theories comply with our knowledge of the sense data, provided we purge our supposed knowledge of all prejudices concerning matter. Also the new logic makes possible theories of continuity and motion which are in harmony with the sense data. This was impossible on the basis of the old logic.

10. And lastly, as Russell has said, "The old logic put thought in fetters, while the new logic gives it wings."[1] The old logic limited the number of hypotheses which could be put forth for the explanation of a fact. It channelized thought and imagination. In this way it put thought in fetters. On the other hand, modern logic enlarges our abstract imagination and makes it more fertile. It provides an infinite number of logically possible hypotheses to be applied in the analysis of any complex fact. The old logic decreed in advance that reality must be of such and such a character. The modern logic makes it possible for us to imagine hypotheses which could not have been imagined otherwise and it makes it possible for us to accept them if they explain the facts. This truely gives wings to thought.

We know that we have not included in this list all of the philosophical importances of mathematical logic, but those which have been listed suffice to indicate that the new logic is one of the great achievements of all ages. It has already born much fruit and it seems to be destined to continue to be fruitful indefinitely.

1. *Our Knowledge of The External World*, P59.

Chapter IX

Summary of The Findings

In this study, we have found that:

1. Russell's logico-mathematical thesis is the contention that logic and pure mathematics form a continuous whole, logic being the beginning of mathematics and mathematics the extention of logic.
2. This thesis dates back to Leibniz and its development has been contributed to chiefly by Boole, Schroeder, Peirce, Frege, Peano, Weierstrass, Dedekind, Cantor, Russell, and Whitehead.
3. Russell, taking advantage of the work of his predecessors and the aid of Whitehead, has developed a new logic in his endeavor to prove this thesis.
4. Russell and Whitehead, making considerable use of Peano's work, have developed a strict logical symbolism which makes it possible to give many logical matters the same precise manipulation as is possible in mathematical operations.
5. Russell, with his new logic, has been able to prove his logico-mathematical thesis, and Russell and Whitehead have been able to give this proof precise demonstration with their new symbolism.
6. Mathematical logic has given mathematics a sound philosophical basis.

7. Mathematical logic has given philosophy a scientific method in much the same way as the work of Galileo and Newton gave science a mathematical method.
8. The new logic has made possible the solution of the problems of the infinitesimal, the infinite, and continuity.
9. The new logic has made possible a solution for Zeno's paradoxes concerning space, time, and motion.
10. The new logic has made it logically possible to construct a philosophy of matter, space, time, and motion in accord with the advances of the new physics.
11. The new logic, by means of the theory of types, has made it possible to solve the paradoxes resulting from the vicious circle fallacy.
12. The new logic refutes the theory of knowledge held by both strict empiricists and idealists and forces us to accept a type of scholastic realism in which universals are admitted to subsist and to be known by *a priori* methods and particulars are admitted to exist and to be known empirically.
13. Modern logic has harmonized logic and sense-data.
14. And, where the old logic put thought in fetters, the new logic gives it wings.

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Vita

(An Autobiography of the Author of this Thesis)

I was born December 29, 1919, at Clarkton Virginia, the third and last son of Wade Hampton and Bessie Calloway Adams. I entered Childrey Grammar School, Clarkton Virginia, in September 1925 and remained in school there until June 1933. In September 1933, I entered Volens High School, Nathalie, Virginia and graduated there in June 1937. In September 1937, I entered Richmond College and graduated there, receiving the Bachelor of Arts degree, June 1941. For the summer sessions of 1941 and 1942, I was enrolled as a graduate student in philosophy at the University of Richmond. In September 1941, I entered The Colgate-Rochester Divinity School, Rochester, New York, where I graduated with the Bachelor of Divinity degree on May 15, 1944.

I was licensed as a minister of religion by Childrey Baptist Church, Clarkton, Virginia, on May 17, 1937, and I was ordained as a Baptist Minister at the same church by the Dan River Baptist Association of Virginia on June 29, 1941.

I was minister of the Deep Run and Ridge Baptist Churches, Henrico County, Virginia, from July 1, 1939 to September 1, 1941, and of the Lakeville Community Congregationalist Church, Lakeville, New York from September 15, 1943 to the present.

I was married to Miss Phyllis Stevenson, Richmond, Virginia, on December 22, 1942.