ROBINS School of Business
University of Richmond
UR Scholarship Repository

Finance Faculty Publications Finance

Summer 2007

A Simple Model of Interest Rate Term Structure

Tom Arnold
University of Richmond, tarnold@richmond.edu

Follow this and additional works at: http://scholarship.richmond.edu/finance-faculty-publications

Part of the Finance and Financial Management Commons, and the Portfolio and Security Analysis Commons

Recommended Citation

Arnold, Tom. "A Simple Model of Interest Rate Term Structure." *Journal of Financial Education* 33 (Summer 2007): 64-72.

This Article is brought to you for free and open access by the Finance at UR Scholarship Repository. It has been accepted for inclusion in Finance Faculty Publications by an authorized administrator of UR Scholarship Repository. For more information, please contact scholarshiprepository@richmond.edu.

A Simple Model of Interest Rate Term Structure for the Classroom

Tom Arnold[1]
University of Richmond

Without much technical expertise, a yield curve model is presented that is very dynamic and can be easily programmed in Excel for classroom presentation or for assignments. By using the output of the model to have students find embedded rates within the yield curve, a discussion of how bond traders speculate on interest rates emerges very easily. Further, the model output can also be used for numerous exercises including the pricing of strips or for evaluating the positions of an entire bond portfolio. Within the exercises, the dynamic nature of the model can be exploited to provide sensitivity analysis.

INTRODUCTION

Bond pricing is generally one of the first applications of time value of money and is introduced with the idea that a constant discount rate is applied to all cash flows. Such a method is fine as an introduction to bond pricing, but quickly collapses when it is revealed that bond yields are implied from the bond price and not the reverse. Further, the implied bond yield is actually a summary number for different discount rates applied to cash flows throughout the life of the bond. Suddenly, something that appears to be a simple exercise requiring an annuity equation becomes much more complex in reality. The goal of this paper is to "decompose" this complex reality and relate it back to the initial simple construct of bond pricing. By performing such a task, the student can now view bond pricing like a bond trader and can understand term structure concepts that are basic knowledge for the Chartered Financial Analysts designation.

In section one, a very simple term structure model is built in Excel that can be introduced into the classroom for lecture or used as the basis for an assignment. In section two, the term structure model is used to price bonds with annual coupons and then the "Goal Seek" feature is applied to find the bond yield to maturity. In section three, applications of the term structure are developed that demonstrate imbedded future short-term rates and the pricing of bond "strips". Section four concludes the paper.

A SIMPLE MODEL FOR THE TERM STRUCTURE OF INTEREST RATES

Although there are numerous theories discussed in texts for why the term structure of interest rates has a particular shape, a discussion of these theories is not presented here. Instead, a model is developed based on a user-supplied 30 day interest rate and a user-supplied growth rate that allows the 30 day rate to be appreciated/depreciated to become longer term rates. The model is very simple to generate and very effective as an educational tool.

The yield curve or term structure of interest rates is simply a mapping of the interest rate (in terms of APR (annual percentage rate)) applicable to a cash flow that emerges at a specific moment in the future. For example, the yield on a 60-day Treasury Bill is the interest rate applicable to a (similar in risk) cash flow that is paid 60 days from the present moment in time. By collecting similar yields that apply for different maturities and of the same risk, the entire yield curve is mapped by matching the applicable bond yields to a set of maturities that span the near-term (30 days to one year) and the far term (one to thirty years) of available debt.

Starting with a 30 day rate of 3.00% APR, the APR rate is grown 1.00% annually to produce the applicable APRs for longer term debt. If the maturity under consideration is less than one year, the 30 day rate is grown in this manner: 3.00%*(1 + 1.00%*({maturity – 30} ÷ 360).

$$R_N = R_{30-day} * \left(1 + g * \left[\left(N - 30\right)/360\right]\right) \tag{1}$$

where, R_N is the rate in terms of APR for the maturity under consideration that is less than a year in maturity, $R_{30\text{-}day}$ is the 30 day user-defined parameter in terms of APR, g is the user-defined annual growth rate for the APR rates, and N is the maturity measured in days.

If the maturity under consideration is one year or more, the 30 day rate is grown in this manner: 3.00%*(1 +1.00%*{330/360})*(1 + 1.00%)$^{\text{maturity -1}}$.

$$R_N = R_{30-day} * \left(1 + g * \left[330/360\right]\right) * (1 + g)^{N-1} \tag{2}$$

where, R_N is the rate in terms of APR for the maturity under consideration that is one year or more in maturity, $R_{30\text{-}day}$ is the 30 day user-defined parameter in terms of APR, g is the user-defined annual growth rate for the APR rates, and N is the maturity measured in years.

Table 1. The Forecasted Term Structure of Interest Rates

Maturity:	*Rate as an APR:*	*Calculation:*
30-Day	3.0000%	Parameter Input
60-Day	3.0025%	3.00%*(1 + 1.00%*({60 – 30}/360))
90-Day	3.0050%	3.00%*(1 + 1.00%*({90 – 30}/360))
180-Day	3.0125%	3.00%*(1 + 1.00%*({180 – 30}/360))
270-Day	3.0200%	3.00%*(1 + 1.00%*({270 – 30}/360))
1 Year	3.0275%	$3.00\%*(1 + 1.00\%*(330/360))*(1 + 1.00\%)^{1-1}$
2 Year	3.0578%	$3.00\%*(1 + 1.00\%*(330/360))*(1 + 1.00\%)^{2-1}$
3 Year	3.0884%	$3.00\%*(1 + 1.00\%*(330/360))*(1 + 1.00\%)^{3-1}$
4 Year	3.1192%	$3.00\%*(1 + 1.00\%*(330/360))*(1 + 1.00\%)^{4-1}$
5 Year	3.1504%	$3.00\%*(1 + 1.00\%*(330/360))*(1 + 1.00\%)^{5-1}$

30-day rate is 3.00% and the annual APR growth rate is 1.00%

Table 1 displays the forecasted rates based on equations (1) and (2) with a 30-day rate of 3.00% and an annual APR growth rate of 1.00%. Table 1 is limited to five years for brevity purposes and can be extended very easily.

To get more out of the simple yield curve model, an Excel spreadsheet is developed which will allow the user to change parameters easily to illustrate different dynamics about the yield curve (see Figure 1).

The programming is not too complex and changing cell B2 in Figure 1 allows the instructor to make the term structure upward sloping ($g > 0$), downward sloping ($g < 0$), or flat ($g = 0$). Unfortunately, Excel's graphing capabilities prevents generating a particularly appealing graph of the yield curve (the interested reader can follow Chapter 1 of Holden (2005)).

In the next section, bond pricing is introduced to the yield curve to illustrate how the bond yield to maturity is more complex than a single number.

THE COMPLEXITY OF BOND PRICING

The first aspect of bond pricing that should be brought to the student's attention is that a bond has cash flows that are discounted based on the individual rates available on the yield curve for each cash flow and not a single rate for all cash flows (like the assumption for using an annuity to price the bond). When the yield curve is flat ($g = 0$ in the model), bond pricing has the appearance of a single rate discounting all cash flows. However, perfectly flat yield curves are unlikely in reality except when debt maturities exceed a certain future point in time.

To illustrate this concept, the Excel spreadsheet in Figure 1 is altered to accommodate a five-year bond with annual coupons (see Figure 2).

Figure 1. Yield Curve Model in Excel

	A	B
1	30-Day Rate (APR):	3.00%
2	APR Growth Rate:	1.00%
3		
4		
5		
6		
7	Maturity in Years:	APR:
8	0.08 {= 30/360}	3.0000% {=B1}
9	0.17 {= 60/360}	3.0025% {=IF(A9<1,$B$1*(1+$B$2*(A9 – 30/360)),$B$1*(1+$B$2*(330/360))*(1+$B$2)^(A9-1)}
10	0.25 {= 90/360}	3.0050% {Copy Cell B9}
11	0.50 {=180/360}	3.0125% {Copy Cell B9}
12	0.75 {=270/360}	3.0200% {Copy Cell B9}
13	1.00	3.0275% {Copy Cell B9}
14	2.00	3.0578% {Copy Cell B9}
15	3.00	3.0884% {Copy Cell B9}
16	4.00	3.1192% {Copy Cell B9}
17	5.00	3.1504% {Copy Cell B9}

Excel formulas in braces. Note: the yield curve ends at five years for brevity...the yield curve can be extended beyond five years very easily.

The bond is priced based on the individual cash flows being discounted at the rate associated with the point in time the cash flow appears. In a sense, each cash flow is evaluated as an individual zero coupon bond. Given this information of multiple rates being applied to pricing the bond, where does the yield to maturity come from? The yield to maturity is the "implied" rate that discounts all of the cash flows to generate the current price of the bond. How does one find it?

A direct solution for the yield to maturity does not exist, but one can iterate to a solution very easily using Excel's "Goal Seek" feature. Figure 2 is amended to include the necessary components for performing the Goal Seek iteration (See Figure 3).
To find the yield to maturity:

- Go to "Goal Seek" under the "Tools" menu
- The "Set Value" parameter in Goal Seek is cell D4
- The "To Value" parameter in Goal Seeks is zero, i.e. the bond price based on the yield to maturity is to be set equal to the bond price based on the yield curve.
- The "By Changing Cell Value" parameter in Goal Seek is cell D2
- Click "Okay" and click "Okay" again after the iteration is completed.

Figure 2. Yield Curve Model in Excel with Bond Pricing

	A	B	C	D
1	30-Day Rate (APR):	3.00%	Bond Price:	$ 1,038.96[a]
2	APR Growth Rate:	1.00%		
3	Coupons (annual):	4%		
4	Par:	$1,000.00		
5	Maturity (years):	5		
6			Bond	Discounted
7	Maturity in Years:	APR:	Cash Flow:	Cash Flow:
8	0.08	3.0000%	$ -[b]	$ -[c]
9	0.17	3.0025%	$ -	$ -
10	0.25	3.0050%	$ -	$ -
11	0.50	3.0125%	$ -	$ -
12	0.75	3.0200%	$ -	$ -
13	1.00	3.0275%	$ 40.00	$ 38.82
14	2.00	3.0578%	$ 40.00	$ 37.66
15	3.00	3.0884%	$ 40.00	$ 36.51
16	4.00	3.1192%	$ 40.00	$ 35.38
17	5.00	3.1504%	$ 1,040.00	$ 890.59

[a] Cell Formula: =SUM(D8:D17)

[b] Cell Formula: =IF(AND(A8 >= 1, A8 < $B$5), $B$3*$B4$, IF(A8 = $B$5, (1 + $B$3)*$B$4, 0))...copy to cells C8 through C17

[c] Cell Formula: =C8/(1+B8)^A8...copy to cells D8 through D17

Note: the spreadsheet is designed to accommodate any bond with annual coupons assuming the bond's maturity is not beyond the length of the yield curve.

Figure 3. Yield Curve Model in Excel with Bond Pricing Extended

	A	B	C	D
1	30-Day Rate (APR):	3.00%	Bond Price:	$ 1,038.96
2	APR Growth Rate:	1.00%	Yield to Maturity:	3.00%[a]
3	Coupons (annual):	4%	Bond Price (YTM):	$ 1,045.80[b]
4	Par:	$1,000.00	Goal Seek:	$ (6.83)[c]
5	Maturity (years):	5		

[a] Any initial value will be fine.

[b] Cell Formula: =PV(D2,B5,-B3*B4) + B4/(1 + D2)^B5...standard bond pricing equation using an annuity to value the coupons.

[c] Cell Formula: = D3 – D1

Cell D2 becomes 3.146% which is the yield to maturity for the bond. Because the yield to maturity is set by comparing a simplified version of the bond price with only one discount rate to an actual bond price that incorporates the yield curve, the yield to maturity is a summary calculation of the effect of the yield curve on the bond.

When first teaching bond pricing, the yield to maturity is generally given to the student to find the bond price. This is not a poor technique for introducing bond pricing, but if the student is going to advance and be able to think more critically about bonds and term structure, this initial method of teaching bond pricing will need to be connected to the more complex reality of the bond market. The simple yield curve model facilitates illustrating how bonds are actually priced and how the yield to maturity is an implied number from an actual bond price and not an input for finding a bond price.

It should be noted that the student does not need any knowledge as to the mechanics of the yield curve model. However, the instructor now has the ability to illustrate what happens to bond prices as the yield curve slopes upward ($g > 0$), downward ($g < 0$), or is flat ($g = 0$). These concepts are discussed in texts, but are very difficult to digest. With the method proposed here, all aspects of term structure and bond pricing are illustrated. Further, by applying different parameters for the 30 day rate and the APR growth rate, individual assignments based on the yield curve can be created for the students if desired. For example, each student can be generated a yield curve based on a different APR growth rate (or different initial 30-day rate) and asked to price a series of bonds or imply future rates (see next section).

FORECASTING AND SECURITY PRICING BASED ON THE YIELD CURVE

Forecasting future rates embedded within the yield curve is a very common exercise for advanced finance classes. Intuitively, assuming the yield curve is not flat, students are able to grasp that the rate of return changes as the maturity is increased. What the students find difficult is what does this mean for, say, one year rates going forward into the future? If the yield curve slope is upward, these rates appear to be increasing, but there is no real indication of the magnitude within the yield curve (notice, this is not even captured by "g" within the yield curve model).

To illustrate this point, assume \$100.00 can be saved for one year or two years. If the \$100.00 is saved for two years, the \$100.00 will become \$106.21 based on the 2-year APR in Table 1 ($\$106.21 = \$100.00^{*}(1 + 3.0578\%)^{2}$). If the \$100.00 is saved for one year, the \$100.00 will become \$103.03 based on the 1-year APR in Table 1 ($\$103.00 = \$100.00^{*}(1 + 3.0275\%)$). The question becomes what future 1-year rate will allow the \$103.03 to appreciate to \$106.21? This future 1-year rate that needs to be found is the 1-year rate embedded within the yield curve that is applicable one year from today. The solution is 3.088% APR ($[\$106.2091 \div \$103.0275] - 1$). Notice, the future 1-year rate is actually higher than the 2-year APR of 3.0578%. This makes sense, because the rate has to capture the amount of compounding lost by having the investment initially appreciate by the smaller current 1-year APR of 3.0275% rather than the 2-year APR of 3.0578%. Further, comparing the future 1-year APR and the current 1-year APR, according to the yield curve, one can expect the 1-year rate to appreciate by about 2% (not 1% as may be implied incorrectly from "g" in the model). A generic formula for implying future one year rates is:

$$R_{1-year,N} = \frac{(1 + R_{N+1})^{N+1}}{(1 + R_N)^N} - 1 \tag{3}$$

where, $R_{1\text{-}year,N}$ is the 1-year APR applicable at time "N" and R_j is the APR on the yield curve at time "j".

One can make an argument that being able to forecast future rates from the yield curve is important for the consideration of future projects. Further, it is very easy to imply the future 1-year APR rates within the yield curve model (see Figure 4).

However, there is a larger more interesting issue available to motivate the topic. Suppose you believe that interest rates are not going to increase very much in the near future and in fact, a future 1-year rate of 3.088% APR is rather high given your perspective. How can you take advantage of your knowledge? In a sense, you desire to only invest in the future 1-year APR rate embedded in the yield curve. If you take a long position in 2-year securities (i.e. receive the 2-year APR) and take a short position in 1-year securities (i.e. pay the 1-year APR), you will effectively isolate the future 1-year APR of 3.088% for yourself. Now the student begins to understand one of the many ways bond traders speculate on interest rates and the task of finding embedded rates takes on a whole new meaning. In fact, one can speculate on any M-year APR rate based on a more general form of equation (3).

$$R_{M-year,N} = \left[\frac{(1 + R_{N+M})^{N+M}}{(1 + R_N)^N} \right]^{\frac{1}{M}} - 1 \tag{4}$$

where, $R_{M\text{-}year,N}$ is the M-year APR applicable at time "N" and R_j is the APR on the yield curve at time "j".

Another way a bond trader can speculate on interest rates is to purchase a bond and sell off some of its component cash flows. One very common decomposition of a bond is to break the bond into a par payment component and a component containing all of the coupon payments (this second part is commonly referred to as a "strip").

Figure 4. Yield Curve Model in Excel with Implied Future Rates

	A	B	C	D
1	30-Day Rate (APR):	3.00%		
2	APR Growth Rate:	1.00%		
3				
4				
5				
6			Future	
7	Maturity in Years:	APR:	1-Year APRs:	
8	0.08	3.0000%		
9	0.17	3.0025%		
10	0.25	3.0050%		
11	0.50	3.0125%		
12	0.75	3.0200%		
13	1.00	3.0275%	3.088%[a]	
14	2.00	3.0578%	3.150%	
15	3.00	3.0884%	3.212%	
16	4.00	3.1192%	3.275%	
17	5.00	3.1504%	3.340%[b]	

[a] Cell Formula: =((1 + B14)^A14)/(1 + B13)^A13 - 1

[b] Based on 3.1819% APR for Year 6...again, it is very easy to extend the yield curve beyond five years.

Further, there is no barrier to selling the individual coupons as separate securities. Given the long-short strategy discussed above, any bond (assuming sufficient maturity) can have portions of its payment stream bought and sold to "lock-in" any particular interest rate on the yield curve. Following this logic and expanding to other debt securities (e.g. mortgages), it is not difficult to segue into a discussion of pass-through securities or topics in financial engineering.

Based on the yield curve model, assignments/lectures can be created to price strips [change the cell formula in footnote "b" of Figure 2 to: =IF(AND(A8 >= 1, A8 <= $B$5), $B$3*$B$4, 0)], or to evaluate investment positions at different points in the yield curve and then evaluate the effects of changing the yield curve by altering the APR growth rate parameter. One can also judge the ability of duration matching to hedge interest rate changes by adopting the duration matching technique in a bond portfolio and then allowing the yield curve to change (via altering the APR growth rate). Again, the simple yield curve model provides the instructor a basis for any number of exercises that exploit yield curve dynamics.

CONCLUSION

Although bond yield to maturity and yield curve dynamics tend to be presented separately within the classroom, there is no doubt that the two subjects are connected in reality. By developing a simple two parameter model for the yield curve, the instructor can now explain directly how the two subjects are connected: the yield to maturity is a summary calculation for the effect of the yield curve on the cash flows of the bond. Further, the output of the model can be exploited for finding embedded rates and for explaining how bond traders speculate on these same embedded rates. Many other exercises, such as pricing strips, can also benefit from the output of the model.

By making the model simple and providing the associated Excel code (also very basic), this paper provides the instructor (no matter what level of computer expertise) with a very powerful tool for developing lectures and assignments at a basic or more advanced level. Although the assignment and lecture suggestions within the paper tend towards a course in investments or fixed income, there really are no limits for how the model or the model output can be applied in the classroom.

ENDNOTES

[1]This article is dedicated in the memory of my friend and colleague, Richard Grayson.

REFERENCES

Holden, C., 2005, *Excel Modeling in Investments,* 2nd Edition, Pearson Prentice Hall. Upper Saddle River, New Jersey.