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PARTIALLY CONFLUENT MAPS AND n-ODS

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Abstract. Let $f: X \to Y$ be a map between topological spaces. A W_f-set in Y is a continuum in Y which is the image under f of a continuum in X. The map f is partially confluent if each continuum in Y is the union of a finite number of W_f-sets, and n-partially confluent if each continuum in Y is the union of n W_f-sets. In this paper, it is shown that every partially confluent map onto an n-cell is weakly confluent. Also, the relationship between partially confluent maps and continua which do not contain n-ods for some n is explored.

A *continuum* is a compact, connected, separable, metric space, A *map* is a continuous function. If $f: X \to Y$ is a map between topological spaces, then f is *monotone* if, for each continuum K in Y, $f^{-1}(K)$ is a continuum, and f is *confluent* if, for each continuum K in Y, f maps each component of $f^{-1}(K)$ onto K. A *W_f-set* in Y is a continuum in Y which is the image under f of a continuum in X. The map f is *weakly confluent* if each continuum in Y is a W_f-set. The map f is *partially confluent* if each continuum in Y is the union of a finite number of W_f-sets. The map f is *n-partially confluent* if each continuum in Y is the union of n W_f-sets. Thus, each monotone map is confluent, each confluent map is weakly confluent, each weakly confluent map is 1-partially confluent, and each n-partially confluent map is partially confluent.

Each type of mappings listed above places restrictions on the degree to which the function can piece together continua or points in the domain to produce new continua in the range. It is for this reason that these maps do not raise the dimension of some one-dimensional continua. For example, it was shown in [5] that the partially confluent image of a one-dimensional acyclic continuum is one-dimensional.

I. Partially confluent maps onto n-cells. An *arc* is a homeomorphic copy of the unit interval, and a subcontinuum K of a continuum X is a *free arc* in X if K is an arc and the boundary points (or point) of K are end-points of K.

The following lemma is both useful and easy to prove.

I.1 LEMMA. *If K is a free arc in the continuum X and f is a map from a continuum onto X, then K is the union of two or fewer W_f-sets. Moreover, if only one end-point of K is a boundary point of K in X, then K is a W_f-set.*

PROOF: The image under f of each component of $f^{-1}(K)$ contains an end-point of K. So there are W_f-sets in K which are images of components of $f^{-1}(K)$ and which are maximal with respect to containing one of the end-points of K and other points of K. The arc K is the union of two or fewer of these W_f-sets. If only one end-point of K is a boundary point of K, then there is a component C of $f^{-1}(K)$ whose image is maximal with respect to containing the boundary point of K and other points of K. Clearly $f(C) = K$ and K is a W_f-set.

A *fan* is a homeomorphic copy of the subset of the plane

$$F = \left\{(r,\theta)|\theta = \frac{1}{n} \text{ and } 0 \leq r \leq 1 \text{ for } n = 1,2,3,\dots\right\},$$

described in polar coordinates, together with the limit line $\{(r,\theta)|\theta = 0$ and $0 \leq r \leq 1\}$.

I.2 LEMMA. *If K is a fan in the continuum X and f is a partially confluent map from a continuum onto X, then the limit line of K is a W_f-set.*

PROOF: The fan K is the union of a countable collection of arcs which will be referred to as $A(a,b_0), A(a,b_1), A(a,b_2), \dots$, where $A(a,b_0)$ is the limit line of K, and, for each $i > 0$, $A(a,b_i)$ is an arc from the junction point a of K to b_i, a point of order one in K. Since f is partially confluent, K is the union of a finite number of W_f-sets, and there is a W_f-set W in K which contains infinitely many of the b_i's. Since W contains more than one b_i, W contains the junction point a, and if W contains b_i, then W contains all of $A(a,b_i)$. By Lemma I.1, each $A(a,b_i)$ in W is a W_f-set. So, the limit line of K is the sequential limit of W_f-sets, and therefore, is a W_f-set.

I.3 THEOREM. *Every partially confluent map from a continuum onto an n-cell is weakly confluent.*

PROOF: Every map onto a 1-cell is weakly confluent. Suppose f is a partially confluent map from the continuum X onto an n-cell Y for $n \geq 2$. Suppose K is a subcontinuum of Y, then there is a sequence of fans

$\{F_1, F_2, F_3, \dots\}$ in Y whose limit lines converge to K. By Lemma I.2, each limit line is a W_f-set. Since K is the sequential limit of W_f-sets, K is a W_f-set. The map f is weakly confluent since each subcontinuum of Y is a W_f-set.

I.4 THEOREM. *Suppose X is a continuum. If there is a partially confluent map from the continuum X onto a continuum with dimension greater than one, then there is a weakly confluent map from X onto a 2-cell.*

PROOF: Suppose Y is a continuum with dimension greater than one, and f is a map from a continuum X onto Y. Then there is a weakly confluent map g from Y onto a 2-cell [**3**, Theorem I, page 328]. Since $g \cdot f$ is a partially confluent map onto a 2-cell, $g \cdot f$ is a weakly confluent map from X onto Y by Theorem I.3.

Theorem I.5 was proven in [**5**] in Theorems II.6 and II.7. But this proof is short and provides an application of the previous theorem, I.4.

I.5 THEOREM. *Suppose Y is a continuum with dimension greater than 2, and f is a partially confluent map from a continuum X onto Y. Then X contains uncountably many nonhomeomorphic subcontinua and X is not acyclic and one-dimensional.*

PROOF: Note that Theorem I.4 implies that there is a weakly confluent map g from X onto $[0,1] \times [0,1]$.

Consider the collection of continua in $[0,1] \times [0,1]$ called the Waraszkiericz spirals. There are uncountably many of them, and each is homeomorphic to a compactification of the positive reals with remainder equal to a circle.

If Φ is a countable collection of continua, there is a Waraszkiewicz spiral S such that no member of Φ maps onto S [**7**, Theorem 2]. The weak confluence of g implies that, for each Waraszkiewicz spiral S, there is a continuum in X which g maps onto S. Thus X contains an uncountable collection of nonhomeomorphic continua.

If X is acyclic and one-dimensional, then each subcontinuum of X is acyclic [**2**, Theorem 2, page 354], and no subcontinuum of X can be mapped onto a Waraszkiewicz spiral [**1**, page 542] which contradicts the fact that f is weakly confluent.

II. Maps which are n-partially confluent. In this section some older results concerning maps onto and from atriodic continua will be generalized in terms of maps onto and from continua which do not contain an n-od. An *n-od* is a continuum K which has a subcontinuum H such that $K\backslash H$ has at least n components, and a *simple n-od* is the union of n arcs which intersect only at one common end-point. For a proof of the following theorem see [6]. It is a generalization of a theorem of Sorgenfrey concerning atriodic continua.

II.1 THEOREM. *Suppose X is a continuum which is the union of a collection Φ of either a finite or countable number of continua, such that $\cap\Phi \neq \phi$, and such that each continuum in Φ contains a point not in the closure of the union of the other members of Φ. Then, if α is the number of continua in Φ, X contains an α-od.*

Maćkowiak has shown that every map onto an atriodic continuum is 2-partially confluent [4, Theorem 6.12, page 53]. The following theorem relies on Theorem II.1 to produce a much more general result.

II.2 THEOREM. *If n is a positive integer and Y is a continuum which does not contain an n-od and f is a map from a continuum X onto Y, then f is $(n \dot{-} 1) \times (n-2)$-partially confluent.*

The proof of the theorem will follow two lemmas. In these lemmas assume that f, X, and Y are as in the statement of the theorem, and, for each subcontinuum K of Y, let $\xi(K)$ be the collection of all subcontinua E of K such that there is a component C of $f^{-1}(K)$ with $f(C) = E$.

II.3 LEMMA. *If K is a subcontinuum of Y and D is an element of $\xi(K)$, then there is a collection $E_1, E_2, E_3, \ldots, E_j$ of elements of $\xi(K)$, where $j \leq n-2$, such that if E is an element of $\xi(K)$ which intersects D, then $E \subset D \cup (\cup_{i=1}^{j} E_i)$.*

PROOF: Let x be a point in K which is contained in an element of $\xi(K)$ which intersects D. If $C_1, C_2, C_3, \ldots$ is a sequence of components of $f^{-1}(K)$ such that $f(C_i)$ contains x and intersects D, and $f(C_i) \cup D \subset f(C_{i+1}) \cup D$ for each i, then some subsequence of the C_i's converges to a component C of $f^{-1}(K)$, and $f(C_i) \cup D \subset f(C) \cup D$ for each i. So if $\mathcal{C}$ is the set of all components C of $f^{-1}(K)$ such that $f(C)$ contains x and intersects D, and $\mathcal{C}$ is ordered by containment in $f(C) \cup D$ for each C in $\mathcal{C}$, then every chain in $\mathcal{C}$ has a greatest element in $\mathcal{C}$, and by Zorn's Lemma, $\mathcal{C}$ has a maximal element. That is, there is an element E_x of $\xi(K)$ which contains x and intersects D such that if E is any other element of $\xi(K)$ which contains x and intersects D, then $E \cup D$ is contained in $E_x \cup D$.

Suppose there are $n-1$ distinct points $x_1, x_2, x_3, \ldots, x_{n-1}$ each of which is contained in an element of $\xi(K)$ which intersects D, and such that, for $i \neq j$, x_i is not contained in $D \cup E_{x_j}$. Let F be a continuum in X such that $f(F)$ contains D, $E_{x_i} \backslash f(F)$ is not empty for each i, and $f(F)$ is not contained in K. Then, according to Theorem II.1, $D \cup f(F) \cup E_{x_1} \cup E_{x_2} \cup \cdots \cup E_{x_{n-1}}$ contains an n-od. This is a contradiction, and the lemma follows from the fact that there is no such collection of points.

II.4 LEMMA. *If K is a subcontinuum of Y, then there do not exist n pairwise disjoint elements of $\xi(K)$.*

PROOF: Suppose $D_1, D_2, D_3, \ldots, D_n$ is a pairwise disjoint collection of elements of $\xi(K)$. Let $F_1, F_2, F_3, \ldots, F_n$ be continua in X such that $f(F_i) \cap f(F_j) = \emptyset$ for $i \neq j$, and $f(F_i)$ is not contained in K for each i. Then $K \cup f(F_1) \cup f(F_2) \cup \cdots \cup f(F_n)$ contains an n-od. This is a contradiction.

PROOF OF THEOREM II.2: Let K be a subcontinuum of Y. According to Lemma II.4, there is a collection of disjoint elements $D_1, D_2, D_3, \ldots, D_m$ of $\xi(K)$ such that $m \leq n-1$, and such that every element of $\xi(K)$ intersects one of the D_i's. According to Lemma II.3, for each i from 1 to m, there is a collection of elements $E_{i1}, E_{i2}, E_{i3}, \ldots, E_{i\alpha i}$ of $\xi(K)$, where $\alpha_i \leq n-2$, such that every point in K which is contained in an element of $\xi(K)$ which intersects D_i is contained in one of the E_{ij}'s. Clearly, the collection of all the D_i's and the E_{ij}'s is a cover of K consisting of $(n-1) \times (n-2)$ or fewer w_f-sets. Since this can be done for each subcontinuum of Y, the map f is $(n-1) \times (n-2)$-partially confluent.

A continuum will be said to have *bounded branching* if there is a positive integer n such that the continuum does not contain an n-od. A continuum will be said to have *finite branching* if it does not contain an infinite-od. There are simple examples of continua with finite but unbounded branching. Theorem II.2 says that every map onto a continuum with bounded branching is n-partially confluent, and that n is determined by the bound on the branching in the range. The method of proof in Theorem II.2 does not seem to work under the slightly different condition that the range has finite branching. Hence the following question.

II.5 QUESTION: *Is every map onto a continuum with finite branching partially confluent?*

Recall that a continuum is *suslinean* if it does not contain an uncountable pairwise disjoint collection of nondegenerate subcontinua. In [5] the author has shown that if X is a suslinean continuum with finite branching and f is a partially confluent map from X onto a continuum Y, then Y has finite branching. The lemma and theorem that follow establish a companion result for continua with bounded branching; that is, if X is a suslinean continuum with bounded branching and f is an n-partially confluent map from X onto a continuum Y, then Y has bounded branching. The proof differs only slightly from the proof in [5].

II.6 LEMMA. *If n and m are positive integers and f is an n-partially confluent map from a suslinean continuum X, which does not contain an m-od onto a continuum Y, then Y is not a simple $(n \times m)$-od.*

PROOF: Suppose Y is a simple $(n \times m)$-od. Then Y is the union of $n \times m$ arcs which will be specified by maps a_i from the unit interval into Y such that $a_i(0)$ is the common end point for each i from 1 to $m \times n$.

For each α in $(1/4, 1]$, the union of $a_i([0,\alpha])$ from $i = 1$ to $m \times n$ is, by the n-partial confluence of f, the union of n or fewer w_f-sets, and one of these w_f-sets contains at least m of the points $a_i(\alpha)$. Let F'_α be a continuum in X which maps onto that w_f-set, and let I_α be a collection of m positive integers such that $f(F'_\alpha)$ contains $a_i(\alpha)$ for each i in I_α. Note that $f(F'_\alpha)$ does not intersect $a_i((\alpha, 1])$ for each i in I_α. In addition, for each α in $(1/4, 1]$ and j in I_α, let $K'_{\alpha j}$ be a subcontinuum of F'_α which maps onto $a_j([1/4, \alpha])$.

Suppose there is an ϵ in $(1/4, 1)$ such that, for each j in I_ϵ, there is an α_j in $(\epsilon, 1]$ such that j is in $I_{\alpha j}$ and F'_ϵ intersects $K'_{\alpha j j}$. Then there is a contradiction, since F'_ϵ would be the core of an m-od in X. So, for each ϵ in $(1/4, 1)$, there is a $j(\epsilon)$ in I_ϵ such that if δ is in $(\epsilon, 1]$ and $j(\epsilon)$ is in I_δ, then F'_ϵ does not intersect $K'_{\delta j(\epsilon)}$. Since there are only a finite number of integers in all of the index sets I_δ, there is an uncountable set E of numbers in $(1/4, 1)$ such that if ϵ and δ are in E, then $j(\epsilon) = j(\delta)$.

If ϵ and δ are in E, then $K'_{\epsilon j(\epsilon)} \subset F'_\epsilon$ and $F'_\epsilon \cap K'_{\delta j(\epsilon)} = \emptyset$. But $j(\epsilon) = j(\delta)$, so $K'_{\epsilon j(\epsilon)} \cap K'_{\delta j(\delta)} = \emptyset$. Thus $\{K'_{\epsilon j(\epsilon)} | \epsilon \text{ is in } E\}$ is an uncountable pairwise disjoint collection of nondegenerate continua in X. This contradicts the fact that X is suslinean, so Y is not a simple $(m \times n)$-od.

II.7 THEOREM. *If m and n are positive integers and f is an n-partially* *confluent map from a suslinean continuum X which does not contain an m-od, then Y does not contain an $(m \times n)$-od.*

PROOF: If Y contains an $(m \times n)$-od, then it is an easy exercise to construct a weakly confluent map g from Y onto a simple $(m \times n)$-od. The map gf is an n-partially confluent map from X onto a simple $(m \times n)$-od. This contradicts Lemma II.6.

The next theorem merely summarizes some of the results of this section in terms of bounded branching and n-partial confluence.

II.8 THEOREM. *If X is a suslinean continuum with bounded branching and f is a map from X onto a continuum Y, then Y is a suslinean continuum with bounded branching if and only if f is an n-partially confluent map for some positive integer n.*

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