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BERGMAN SPACES ON AN ANNULUS AND THE BACKWARD BERGMAN SHIFT

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ABSTRACT. In this paper, we will give a complete characterization of the invariant subspaces $\mathcal{M}$ (under $f \to zf$) of the Bergman space $L^p_a(G)$, $1 < p < 2$, G an annulus, which contain the constant function 1. As an application of this result, we will characterize the invariant subspaces of the adjoint of multiplication by z on the Dirichlet spaces D_q, $q > 2$, as well as the invariant subspaces of the backward Bergman shift $f \to (f - f(0))/z$ on $L^p_a(\mathbb{D})$, $1 < p < 2$.

1. INTRODUCTION

Let $1 < p < \infty$ and $G = \{z \in \mathbb{C} : 1 < |z| < \rho\}$ be an annulus in the complex plane. Define the *Hardy space* $H^p(G)$ as the space of analytic functions f on G with

$$\sup_{1<r<\rho} \int_{\mathbb{T}} |f(r\zeta)|^p |d\zeta| < \infty. \tag{1.1}$$

It is well known [17] [18] that $H^p(G)$ is a Banach space and that the operator $f \to zf$ is a continuous linear transformation on $H^p(G)$. There has been much study on the so-called *invariant subspaces* of $H^p(G)$, the closed linear manifolds $\mathcal{M}$ of $H^p(G)$ with $z\mathcal{M} \subset \mathcal{M}$. For example, an easy exercise, using the Cauchy integral formula, shows that if $\mathcal{K}$ is an invariant subspace of the Hardy space $H^p(\rho\mathbb{D})$ (the analytic functions on $\rho\mathbb{D} = \{|z| < \rho\}$ with (1.1) finite), then $\mathcal{K}|_G$ is an invariant subspace of $H^p(G)$. The non-zero invariant subspaces of $H^p(\rho\mathbb{D})$ have been characterized by Beurling [4] as $\phi H^p(\rho\mathbb{D})$, where ϕ is a bounded analytic function on $\rho\mathbb{D}$ whose non-tangential boundary values on $\rho\mathbb{T} = \{|z| = \rho\}$ have modulus one almost everywhere. Sarason [18] characterized the *fully invariant* subspaces of $H^p(G)$, those which are invariant under multiplication by any rational function with poles outside the closure of G, as $\phi H^p(G)$, where ϕ is a bounded analytic function on G whose non-tangential boundary values have constant modulus on each component of the boundary of G. Royden [17] characterized the invariant subspaces of $H^p(G)$ which contain the constant function 1 as $\{f \in H^p(G) : \phi f \in H^p(\mathbb{T})\}$, where ϕ is a bounded analytic function on $\mathbb{D}$ whose boundary values have modulus one almost everywhere, and $H^p(\mathbb{T})$ denotes the usual Hardy space on

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the unit circle $\mathbb{T} = \{|z| = 1\}$. (We remark that that the function ϕ used above is often called an "inner function" for $H^p(\rho\mathbb{D})$, $H^p(G)$, $H^p(\mathbb{D})$, respectively.)

A more difficult problem, and the one we focus on in this paper, is the characterization of the invariant subspaces of the *Bergman space* $L^p_a(G)$ of functions which are analytic on G with

$$\int_G |f|^p dA < \infty.$$

Here dA is area measure on G and "invariant" again means invariant under the operator $f \to zf$. In contrast to the Hardy space, Bergman space functions are not, in general, of bounded characteristic (the quotient of two bounded analytic functions on G) and thus useful concepts such as non-tangential boundary values and factorization, which are essential to the characterization of the invariant subspaces in the Hardy space, are not available to us. Also in contrast to the Hardy space, the Bergman space has a rich lattice of invariant subspaces which can be very complicated. For example, as in the Hardy space, $\mathcal{K}|_G$, where $\mathcal{K}$ is an invariant subspace of $L^2_a(\rho\mathbb{D})$, is a closed invariant subspace of $L^2_a(G)$. It is known [3] that given any $n \in \mathbb{N} \cup \{\infty\}$ there is an invariant subspace $\mathcal{K} \subset L^2_a(\rho\mathbb{D})$ such that $\dim(\mathcal{K}/z\mathcal{K}) = n$. (This property is often refered to as the "codimension n" property.) This is in stark contrast to the Hardy space case where every non-zero invariant subspace $\mathcal{K} \subset H^p(\rho\mathbb{D})$ has $\dim(\mathcal{K}/z\mathcal{K}) = 1$. In fact, if one attempts to characterize the fully invariant subspaces of $L^p_a(G)$, as Sarason did for $H^p(G)$, one runs into an equally complicated problem since given any $n \in \mathbb{N} \cup \{\infty\}$, there is a fully invariant subspace $\mathcal{M}$ of $L^2_a(G)$ with $\dim(\mathcal{M}/(z - \lambda)\mathcal{M}) = n$ for all $\lambda \in G$ [15], again indicating the complexity of these subspaces.

In this paper, we will focus our attention on the invariant subspaces $\mathcal{M} \subset L^p_a(G)$ which contain the constant function 1 and obtain, at least for $1 < p < 2$, a result similar to that of Royden for the Hardy space. Surprisingly, certain functions in $\mathcal{M}$ will be of bounded characteristic and thus one has vital tools to work with. As an application of this result, we will give a characterization of the invariant subspaces for the *backward Bergman shift*

$$L : L^p_a(\mathbb{D}) \to L^p_a(\mathbb{D}), \quad Lf = \frac{f - f(0)}{z}$$

for $1 < p < 2$. The invariant subspaces of the backward shift L on the Hardy spaces $H^p(\mathbb{D})$ where characterized in [17] [20].

2. Terminology and Main Theorems

In order to state our theorem, as well as the applications, we need to set some terminology. Throughout this paper, $\mathbb{D}$ will denote the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T}$ will denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. For $1 \leq p \leq \infty$, $H^p(\mathbb{D})$ will denote the usual Hardy space on the disk. We refer the reader to [6] for the basic facts about these functions. In this paper, when we use the term *inner function*, we will mean a function $I \in H^\infty(\mathbb{D})$ whose non-tangential boundary values have modulus equal to one almost everywhere (as opposed to inner functions defined on general domains [17]). An inner function I can be factored as $I = BS_\mu$, where B is a Blaschke product with zeros $\{a_k\}$ repeated according to multiplicity, and S_μ is a singular inner function with positive singular measure μ. We define the set

$$\operatorname{spec}(I) = \operatorname{clos}\{a_k\} \cup \operatorname{supp}(\mu).$$

Moreover, given a closed set $E \subset \mathbb{T}$ and an inner function I satisfying

$$\int_{\mathbb{T}} \log \operatorname{dist}(\zeta, E \cup \operatorname{spec}(I))|d\zeta| > -\infty, \tag{2.1}$$

there exists an outer function $\mathcal{O} \in A^\infty$ (infinitely differentiable in $\overline{\mathbb{D}}$) with

$$\mathcal{O}^{-1}(0) = (E \cup \operatorname{spec}(I)) \cap \mathbb{T}, \quad I\mathcal{O} \in A^\infty.$$

Such functions are important in the study of ideals of analytic functions and were discovered by Korenblum [12], Lemma 23. We will refer to the function $I\mathcal{O}$ as the associated Korenblum function for E and I. We set $\mathbb{D}_e$ to be the extended exterior disk $\mathbb{D}_e = \mathbb{C}_\infty \backslash \overline{\mathbb{D}}$ and for $1 \leq p \leq \infty$ let $H^p(\mathbb{D}_e)$, the Hardy space on $\mathbb{D}_e$, be the analytic functions on $\mathbb{D}_e$ with

$$\sup_{r>1} \int_{\mathbb{T}} |f(r\zeta)|^p \frac{|d\zeta|}{2\pi r} < \infty.$$

We let

$$N(\mathbb{D}_e) = \{\frac{f}{g} : f, g \in H^\infty(\mathbb{D}_e)\}, \qquad N(\mathbb{D}) = \{\frac{f}{g} : f, g \in H^\infty(\mathbb{D})\}$$

denote the functions of bounded characteristic on $\mathbb{D}_e$ and $\mathbb{D}$ respectively, and define

$$N^+(\mathbb{D}) = \{\frac{f}{g} : f, g \in H^\infty(\mathbb{D}), g \text{ outer}\}.$$

Functions in both $N(\mathbb{D})$ and $N(\mathbb{D}_e)$ have nontangential limits a.e. on $\mathbb{T}$ [6], p. 17, and we will say that $G \in N(\mathbb{D})$ is a *pseudo-continuation* of $g \in N(\mathbb{D}_e)$ if these limits are equal a.e. By a theorem of Lusin and Privalov [14], p. 212, pseudo-continuations are unique when they exist. Pseudo-continuations play an important role in other invariant subspace problems [16] [17] [20] and will play an important role here. Every function $f \in L^p_a(G)$ has a Laurent expansion which we write as

$$f(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=-\infty}^{0} a_n z^n$$

and set f_+ to be the first sum and f_- to be the second. We now state our main results.

Theorem 2.1. *Let $1 < p < 2$ and $1 \in \mathcal{M} \neq L^p_a(G)$ be an invariant subspace. Then there is a inner function I and a closed set $E \subset \mathbb{T}$ satisfying condition (2.1) such that $\mathcal{M} = \mathcal{M}(E, I)$, where $\mathcal{M}(E, I)$ is the space of functions $f \in L^p_a(G)$ which satisfy the following two conditions:*

1. *$f_-\Omega \in H^1(\mathbb{D}_e)$, where $\Omega(z) = \overline{(I\mathcal{O})(1/\overline{z})}$ and $I\mathcal{O}$ is the Korenblum function.*
2. *f_- has a pseudo-continuation to $\mathbb{D}$ with $If_- \in N^+(\mathbb{D})$.*

Corollary 2.2. *With the notation above, f_- has an analytic continuation to $\mathbb{C}_\infty \backslash (E \cup spec(I))$.*

Corollary 2.3. *If, in the notation above, $I = BS_\mu$, where B is a finite Blaschke product, then $f \in L^p_a(G)$ belongs to $\mathcal{M}(E, I)$ if and only if the following three conditions are satisfied:*

(1) $f_- \in N(\mathbb{D}_e)$

(2) *f_- has an analytic continuation to $\mathbb{C}_\infty \backslash (E \cup spec(I))$*

(3) $If_- \in N^+(\mathbb{D})$.

Remarks:

(1) In Theorem 2.1, the role of the function Ω will be to control the growth of f_- near $\mathbb{T}$. Notice that the function Ω is not needed in Corollary 2.3.

(2) The hypothesis that $\mathcal{M} \neq L^p_a(G)$ is crucial here since, in general, f_- is not of bounded characteristic.

The main technique here will be to relate the invariant subspaces $\mathcal{M}$ of the Bergman space $L^p_a(G)$, via annihilators and the Cauchy transform, to invariant subspaces of the analytic Dirichlet space D_q, $q = p(p-1)^{-1}$ of functions g which are analytic on the disk $\mathbb{D}$ with

$$\int_{\mathbb{D}} |g'|^q dA < \infty.$$

For $1 < p < 2$, the conjugate index $q > 2$ and the invariant subspaces of D_q are known [21]. We will then reverse this process to transfer this information back to the Bergman space.

One application of this result, we be the characterization of the invariant subspaces for the adjoint of multiplication by z on the Dirichlet spaces D_q for $q > 2$. Another application will be the characterization of the invariant subspaces of the *backward Bergman shift*

$$L : L^p_a(\mathbb{D}) \to L^p_a(\mathbb{D}), \quad Lf = \frac{f - f(0)}{z}$$

for $1 < p < 2$. For $1 < p < \infty$ it is known [17] [20] that every invariant subspace of the backward shift L on the Hardy space $H^p(\mathbb{D})$ is of the form $\mathcal{K}_\phi$, for some inner function ϕ, where $\mathcal{K}_\phi$ is the space of $f \in H^p(\mathbb{D})$ such that $f(\frac{1}{z})$ has a pseudo-conntinuation to $\mathbb{D}$ with $\phi f(\frac{1}{z}) \in zH^p(\mathbb{D})$. Our main theorem for the backward Bergman shift is as follows:

Theorem 2.4. *Let $1 < p < 2$ and $\mathcal{K} \subset L^p_a(\mathbb{D})$ be a non-trivial ($\mathcal{K} \neq L^p_a(\mathbb{D})$) L-invariant subspace. Then there is a closed set $E \subset \mathbb{T}$ and an inner function I satisfying (2.1) such that $\mathcal{K} = \mathcal{K}(E, I)$, where $\mathcal{K}(E, I)$ is the space of functions $f \in L^p_a(\mathbb{D})$ which satisfy the following two conditions:*

(1) $f(\frac{1}{z})\Omega \in H^1(\mathbb{D}_e)$

(2) *$f(\frac{1}{z})$ has a pseudo-continuation to $\mathbb{D}$ with $If(\frac{1}{z}) \in zN^+(\mathbb{D})$.*

Moreover $f(\frac{1}{z})$ *has an analytic continuation to* $\mathbb{C}_\infty \setminus (E \cup spec(I))$. *In addition if* $I = BS_\mu$, *where* B *is a finite Blaschke product, then* $f \in L^p_a(\mathbb{D})$ *belongs to* $\mathcal{K}(E, I)$ *if and only if* f *satisfies the following three conditions:*

(1) $f(\frac{1}{z}) \in N(\mathbb{D}_e)$.
(2) $f(\frac{1}{z})$ *has a analytic continuation to* $\mathbb{C}_\infty \setminus (E \cup spec(I))$.
(3) $If(\frac{1}{z}) \in zN^+(\mathbb{D})$.

Using this theorem, we can determine (see Section 7) which vectors $f \in L^p_a(\mathbb{D})$, $1 < p < 2$, are *cyclic* for L, that is

$$\operatorname{span}_{L^p_a}\{f, Lf, L^2 f, \cdots\} = L^p_a(\mathbb{D}).$$

Note that in the $H^p(\mathbb{D})$ case this question was answered in [20].

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3. Sobolev spaces

A key step in our proof will be the examination of certain Sobolev spaces which arise from our z-invariant subspaces of the Bergman space. This technique of relating Bergman space problems to Sobolev space problems was first employed by Havin [9] in his investigations of polynomial and rational approximation in the Bergman space. We begin with some basic notation and facts about Sobolev spaces and refer the reader to [1] for further information. For $1 < q < \infty$, define the *Sobolev space* $W^q_1 = W^q_1(\mathbb{C})$ as the space of complex valued functions f for which

$$\|f\|_q = \|f\|_{L^q} + \|\nabla f\|_{L^q} < \infty,$$

with ∇f taken in the sense of distributions. For a bounded open set $U \subset \mathbb{C}$, let $W^{q,0}_1(U)$ be the closure of $C^\infty_0(U)$ (infinitely differentiable functions with compact support in U) in the Sobolev norm. By the Poincaré inequality, an equivalent norm on $W^{q,0}_1(U)$ is

$$\|f\|_{q,0} = \|\nabla f\|_{L^q}. \tag{3.1}$$

By the Sobolev imbedding theorem [1], p. 97, $W^{q,0}_1(U)$, $q > 2$, is a Banach algebra of continuous functions on $\mathbb{C}$ (in fact they are Holder continuous with exponent $1 - 2/q$). In fact, $W^{q,0}_1(U)$ can be described in terms of its zero set by the following result [2]:

Proposition 3.1 (Bagby). *For* $q > 2$, $W^{q,0}_1(U) = \{f \in W^q_1 : f|_{\mathbb{C}\setminus U} = 0\}$.

For $1 < q \leq 2$ there is an analogous result to describe $W^{q,0}_1(U)$ which involves capacity. For $q > 2$, capacity is not needed since the functions are continuous.

For $1 < p < \infty$, the dual of $L^p(G) = L^p(G, dA)$ can be identified with $L^q(G)$, $q = p(p-1)^{-1}$, by the bi-linear pairing

$$(3.2) \qquad < f, g > \stackrel{def}{=} \int_G fg dA.$$

We will now relate the annihilator of the Bergman space with the Sobolev space. If $f \in (L_a^p(\rho\mathbb{D})|_G)^\perp$, this is $f \in L^q(G)$ with

$$\int_G \frac{f}{z-w} dA = 0 \quad \forall |w| > \rho,$$

the Calderon-Zygmund theory [22], p. 39, 60, says the Cauchy transform

$$(Cf)(w) \stackrel{def}{=} -\frac{1}{\pi} \int_G \frac{f}{z-w} dA$$

belongs to L^q with

$$(3.3) \quad \overline{\partial} C(f) = \frac{1}{2}(\partial/\partial_x + i\partial/\partial_y)Cf = f \quad \text{and} \quad \|\nabla Cf\|_{L^q} \leq K\|\overline{\partial} Cf\|_{L^q} = K\|f\|_{L^q},$$

(for some $K > 0$ independent of f), from which $Cf \in W_1^q$. Since Cf vanishes off $\rho\mathbb{D}$, then by Proposition 3.1, $Cf \in W_1^{q,0}(\rho\mathbb{D})$. Furthermore, since the support of the measure fdA is in $\overline{G}$, then Cf is analytic in $\mathbb{D}$ and hence belongs to

$$W_{1,A}^{q,0}(\rho\mathbb{D}) \stackrel{def}{=} \{f \in W_1^{q,0}(\rho\mathbb{D}) : \text{f is analytic on } \mathbb{D}\}.$$

Remark: Given any analytic function $g \in L^q(\mathbb{D})$ with

$$(3.4) \qquad \int_{\mathbb{D}} |\nabla g|^q dA < \infty,$$

one can extend g (by reflection and then making g zero near $\rho\mathbb{T}$) to a $\tilde{g} \in W_{1,A}^{q,0}(\rho\mathbb{D})$. We will make use of this observation several times in this paper.

For $g \in W_{1,A}^{q,0}(\rho\mathbb{D})$ one can use Green's formula to show that

$$g(\lambda) = -\frac{1}{\pi} \int_{\rho\mathbb{D}} \frac{\overline{\partial} g}{z-\lambda} dA.$$

Since $\overline{\partial} g = 0$ on $\mathbb{D}$, then

$$(3.5) \qquad g(\lambda) = -\frac{1}{\pi} \int_G \frac{\overline{\partial} g}{z-\lambda} dA = C(\overline{\partial} g)(\lambda).$$

Moreover if p is an analytic polynomial, then $pg \in W_{1,A}^{q,0}(\rho\mathbb{D})$ and

$$(3.6) \qquad pC(\overline{\partial} g) = pg = C(\overline{\partial}(pg)) = C(p\overline{\partial} g).$$

If $f \in L_a^p(G)^\perp$, that is $f \in L^q(G)$ with

$$\int_G \frac{f}{z-w} dA = 0 \quad \forall |w| < 1, |w| > \rho,$$

then Cf vanishes off G and so by Proposition 3.1, $Cf \in W_1^{q,0}(G)$.

This next result is due to Havin [9].

Lemma 3.2 (Havin). *Let U be a bounded open set and $1 < p < \infty$. Then $f \in L^q(U)$ satisfies*

$$\int_U u f dA = 0 \ \ \forall u \in L^p_a(U)$$

if and only if there is an $F \in W_1^{q,0}(U)$ with $\overline{\partial} F = f$.

Proposition 3.3. *The maps*

$$C : (L^p_a(\rho\mathbb{D})|_G)^{\perp} \to W^{q,0}_{1,A}(\rho\mathbb{D})$$

$$C : L^p_a(G)^{\perp} \to W^{q,0}_1(G)$$

are continuous, invertible, linear operators.

Proof. First notice, from the above remarks, that C is linear and well defined on the appropriate spaces. Then notice from (3.3), that for f in either $(L^p_a(\rho\mathbb{D})|_G)^{\perp}$ or $L^p_a(G)^{\perp}$,

$$\|f\|_{L^q} = \|\overline{\partial} C f\|_{L^q} \leq \|\nabla C f\|_{L^q} \leq K \|\overline{\partial} C f\|_{L^q} = K\|f\|_{L^q}.$$

Thus, by the Poincaré inequality (3.1), C is both continuous and bounded below on both spaces. To finish, we just need to show that C is surjective on each space.

Given $g \in W^{q,0}_1(G)$ we use Lemma 3.2 to get $\overline{\partial} g \in L^p_a(G)^{\perp}$. By (3.5), $C(\overline{\partial} g) = g$, and thus $C : L^p_a(G)^{\perp} \to W^{q,0}_1(G)$ is surjective. For the other map, we let $g \in W^{q,0}_{1,A}(\rho\mathbb{D})$ and apply Havin's lemma again to get

$$\int_{\rho\mathbb{D}} h \overline{\partial} g dA = 0 \quad \forall h \in L^p_a(\rho\mathbb{D}).$$

But since $\overline{\partial} g = 0$ on $\mathbb{D}$, then $\overline{\partial} g \in (L^p_a(\rho\mathbb{D})|_G)^{\perp}$. Again notice $C(\overline{\partial} g) = g$ and so we have shown $C : (L^p_a(\rho\mathbb{D})|_G)^{\perp} \to W^{q,0}_{1,A}(\rho\mathbb{D})$ is surjective. □

Also needed later will be this standard fact from trace theory in Sobolev spaces [13].

Proposition 3.4. *Let $1 < p < \infty$ and $f \in W^{p,0}_1(\rho\mathbb{D})$. Then*

$$\lim_{r \to 1^-} f(r\zeta) \quad \textit{and} \quad \lim_{r \to 1^+} f(r\zeta)$$

exist almost everywhere $|d\zeta|$ as well as in $L^p(\mathbb{T}, |d\zeta|)$ and they are equal a.e.

4. Correspondence with the Dirichlet space

For $1 < q < \infty$ define the *L^q-Dirichlet space* D_q as the space of analytic functions on $\mathbb{D}$ with

$$\int_{\mathbb{D}} |f'|^q dA < \infty.$$

D_q becomes a Banach space when given the norm

$$\|f\|_{D_q} = |f(0)| + \Big(\int_{\mathbb{D}} |f'|^q dA \Big)^{1/q}.$$

For $q > 2$, the Sobolev imbedding theorem says that D_q is a Banach algebra of continuous functions on $\overline{\mathbb{D}}$.

Remark: D_q can also be thought of as the analytic extensions (via the Poisson integral) of $f(\zeta) \in H^q(\mathbb{T})$ with norm

$$\|f\| = \int_{\mathbb{T}} |f(\zeta)|^q |d\zeta| + \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{f(\zeta) - f(\xi)}{\zeta - \xi} \right|^q |d\zeta||d\xi|.$$

In fact, these two norms are equivalent [22], Chapter 5, Section 5. This space of functions is often called the Besov space and we bring this to the readers attention since Shirokov [21] uses this terminology in his paper.

The operator $f \to zf$ is continuous on D_q and one can ask about the invariant subspaces (under multiplication by z) of D_q. For $q > 2$, we use the fact that D_q is a Banach algebra, along with the density of the polynomials in D_q, to see that the invariant subspaces are precisely the closed ideals of D_q (equivalently the Beosv space) and have been characterized by Shirokov [21] as follows:

Theorem 4.1 (Shirokov). *Given any (closed) ideal $\mathcal{F} \subset D_q$, $(q > 2)$, there is a closed set $E \subset \mathbb{T}$ and an inner function I such that*

$$\mathcal{F} = \mathcal{F}(E, I) \stackrel{def}{=} \{f \in D_q : f|_E = 0, \frac{f}{I} \in H^\infty\}.$$

Remarks:

(1) For a general closed set $E \subset \mathbb{T}$ and inner function I, the ideal $\mathcal{F}(E, I)$ might be zero. Using the fact that $D_q \subset \mathrm{Lip}_{1-2/q}(\overline{\mathbb{D}})$, for $q > 2$, along with [5], Theorem 1, [23], and [24], Corollary 4.8, we see that $\mathcal{F}(E, I)$ is non-zero if and only if condition (2.1) is satisfied. Condition (2.1) determines the triviality of other ideals of analytic functions, see for example [12], p. 113.

(2) When $I = 1$ (or just a finite Blaschke product) then (2.1) is equivalent to the well-known Beurling-Carleson condition [5]

$$|E| = 0 \quad \text{and} \quad \sum_n |I_n| \log |I_n| > -\infty,$$

where $\{I_n\}$ are the complimentary arcs of E. Such E are often called *Carleson sets.*

(3) The set E is the common zeros of the ideal on $\mathbb{T}$ and the inner function I is the greatest common divisor of the inner divisors of functions in the ideal. Thus if $\mathcal{F}(E,I) \neq 0$ and $I\mathcal{O}$ is the associated Korenblum function for E and I, then

$$\operatorname{span}_{D_q}\{z^n I\mathcal{O} : n = 0,1,2,\cdots\} = \mathcal{F}(E,I). \tag{4.1}$$

To relate the Dirichlet space with the Bergman space, we proceed as follows: Recall that for $1 < p < 2$ we have

$$1 \in \mathcal{M} \subset L^p_a(G).$$

By the invariance of $\mathcal{M}$ and the density of polynomials in $L^p_a(\rho\mathbb{D})$, we obtain

$$L^p_a(\rho\mathbb{D})|_G \subset \mathcal{M} \subset L^p_a(G).$$

Taking annihilators we obtain

$$L^p_a(G)^\perp \subset \mathcal{M}^\perp \subset (L^p_a(\rho\mathbb{D})|_G)^\perp$$

with, by our bi-linear pairing (3.2), $z\mathcal{M}^\perp \subset \mathcal{M}^\perp$. Now apply the Cauchy transform along with Proposition 3.3 to get

$$W^{q,0}_1(G) \subset C\mathcal{M}^\perp \subset W^{q,0}_{1,A}(\rho\mathbb{D})$$

with, by (3.6), $zC\mathcal{M}^\perp \subset C\mathcal{M}^\perp$. Note also that since $1 < p < 2$, then $q > 2$.

Lemma 4.2. *$C\mathcal{M}^\perp = \{f \in W^{q,0}_{1,A}(\rho\mathbb{D}) : f|_{\mathbb{D}} \in \mathcal{F}\}$, where $\mathcal{F}$ is a closed ideal of D_q.*

Proof. The map

$$\Phi : W^{q,0}_{1,A}(\rho\mathbb{D}) \to D_q, \quad \Phi(f) = f|_{\mathbb{D}}$$

is a continuous, surjective (by (3.4)), linear transformation with, by Proposition 3.1, $\ker(\Phi) = W^{q,0}_1(G)$. Thus Φ induces the continuous invertible transformation

$$\tilde{\Phi} : W^{q,0}_{1,A}(\rho\mathbb{D})/W^{q,0}_1(G) \to D_q, \quad \tilde{\Phi}([f]) = f|_{\mathbb{D}},$$

where $[f]$ is a coset of the quotient space $W^{q,0}_{1,A}(\rho\mathbb{D})/W^{q,0}_1(G)$. The operator T_z, multiplication by z on $W^{q,0}_{1,A}(\rho\mathbb{D})$, induces

$$\tilde{T}_z : W^{q,0}_{1,A}(\rho\mathbb{D})/W^{q,0}_1(G) \to W^{q,0}_{1,A}(\rho\mathbb{D})/W^{q,0}_1(G), \quad \tilde{T}_z([f]) = [zf],$$

with

$$M_z\tilde{\Phi} = \tilde{\Phi}\tilde{T}_z, \tag{4.2}$$

where M_z is multiplication by z on D_q. Thus if $W^{q,0}_1(G) \subset C\mathcal{M}^\perp \subset W^{q,0}_{1,A}(\rho\mathbb{D})$ is T_z-invariant, then $[C\mathcal{M}^\perp]$ is $\tilde{T}_z$-invariant and so by (4.2), $[C\mathcal{M}^\perp] = \tilde{\Phi}^{-1}(\mathcal{F})$, where $\mathcal{F}$ is an invariant subspace of D_q. Thus $C\mathcal{M}^\perp = \{h \in W^{q,0}_{1,A}(\rho\mathbb{D}) : h|_{\mathbb{D}} \in \mathcal{F}\}$. $\square$

By Lemma 4.2 and Theorem 4.1, there is a closed set $E \subset \mathbb{T}$ and an inner function I with

$$C\mathcal{M}^{\perp} = \{h \in W^{q,0}_{1,A}(\rho\mathbb{D}) : h|_{\mathbb{D}} \in \mathcal{F}(E,I)\}.$$

Now use the fact $C^{-1} = \overline{\partial}$, with (3.5), along with the Hahn-Banach theorem to get

$$\mathcal{M} = \mathcal{M}(E,I) \stackrel{def}{=} (\overline{\partial}\{h \in W^{q,0}_{1,A}(\rho\mathbb{D}) : h|_{\mathbb{D}} \in \mathcal{F}(E,I)\})^{\perp}. \tag{4.3}$$

Moreover, notice from Proposition 3.1 that $\mathcal{M}(E,I) \neq L^p_a(G)$ if and only if $\mathcal{F}(E,I) \neq 0$. Thus $\mathcal{M}(E,I) \neq L^p_a(G)$ if and only if condition (2.1) holds. The rest of the paper is now dedicated to describing $\mathcal{M}(E,I)$.

5. Pseudo-continuations and analytic continuations

Recall that for $1 < p < \infty$, every invariant subspace $1 \in \mathcal{K} \subset H^p(G)$ is of the form

$$\mathcal{K}_I = \{f \in H^p(G) : If \in H^p(\mathbb{T})\},$$

for some inner function I. Since If is the boundary function for some $H^p(\mathbb{D})$ function, then

$$(If)(z) = h(z), \quad |z| < 1, \; h \in H^p(\mathbb{D}).$$

Thus if we define the function $\tilde{f}$ on $\mathbb{D}\backslash\text{spec}(I)$ by $\tilde{f} = h/I$, then $\tilde{f}$ is a pseudo-continuation of f. Moreover, if J is an arc in $\mathbb{T}$ disjoint from spec(I), then there is a neighborhood U of J for which I is bounded away from zero on $U \cap \{|z| < 1\}$. Thus since f is H^p near J, as is $\tilde{f}$, and $f(\zeta) = \tilde{f}(\zeta)$ a.e., then $\tilde{f}$ is an analytic continuation of f across J [20], p. 41. Thus every $f \in \mathcal{K}_I$ has an analytic continuation $\tilde{f}$ to $\rho\mathbb{D}\backslash\text{spec}(I)$ with $I\tilde{f} \in H^p(\mathbb{D})$.

For $1 < p < 2$, every invariant subspace $1 \in \mathcal{M} \subset L^p_a(G)$ is of the form $\mathcal{M} = \mathcal{M}(E,I)$. As a first step in describing $\mathcal{M}(E,I)$, we will prove a pseudo-continuation and analytic continuation result similar to that of the Hardy space case. Our result is the following:

Proposition 5.1. *Let $1 < p < 2$ and $f \in \mathcal{M}(E,I) \neq L^p_a(G)$, then $f_- \in \mathcal{M}(E,I)$ and if $\Omega(z) = \overline{(I\mathcal{O})(1/\overline{z})}$, where $I\mathcal{O}$ is the associated Korenblum function, then*

1. *$f_-\Omega \in H^1(\mathbb{D}_e)$.*
2. *f_- has a pseudo-continuation to $\mathbb{D}$.*
3. *$If_- \in N^+(\mathbb{D})$.*

To prove Proposition 5.1 we need a preliminary lemma.

Lemma 5.2. *For $\lambda \in G$ define $L_\lambda : L^p_a(G) \to L^p_a(G)$*

$$L_\lambda f = \frac{f - f(\lambda)}{z - \lambda}.$$

Then $L_\lambda\mathcal{M}(E,I) \subset \mathcal{M}(E,I)$ for all $\lambda \in G$.

Proof. By (4.3), $C\mathcal{M}(E,I)^{\perp} = \{h \in W_1^{q,0}(\rho\mathbb{D}) : h|_{\mathbb{D}} \in \mathcal{F}(E,I)\}$. From (4.1) $I\mathcal{O} \in A^{\infty}$ with

$$\text{span}_{D_q}\{z^n I\mathcal{O} : n = 0,1,2,\cdots\} = \mathcal{F}(E,I). \tag{5.1}$$

Now define h_0 on $\rho\mathbb{D}$ as follows (For the purpose of this construction, we will assume $\rho > 2$.):

$$h_0(z) = \begin{cases} (I\mathcal{O})(z) & |z| \le 1 \\ (I\mathcal{O})(\frac{1}{\bar{z}}) & 1 \le |z| \le \frac{1}{2}\rho \\ k(z) & \frac{1}{2}\rho \le |z| \le \rho \end{cases} \tag{5.2}$$

where $k(z)$ is a smooth function which vanishes near $\rho\mathbb{T}$ and agrees with $(I\mathcal{O})(1/\bar{z})$ on $\frac{1}{2}\rho\mathbb{T}$. The function $h_0 \in W_{1,A}^{q,0}(\rho\mathbb{D})$ with $\bar{\partial}h_0 \in L^{\infty}$ (since $h_0 \in \text{Lip}_1$ and thus $\nabla h_0 \in L^{\infty}$).

Moreover, by (4.3), (5.1), and Lemma 4.2,

$$C\mathcal{M}(E,I)^{\perp} = \text{span}_{W_1^q}\{z^n h_0 : n \ge 0\} \bigvee W_1^{q,0}(G)$$

and so by (3.3)

$$\mathcal{M}(E,I)^{\perp} = \text{span}_{L^q}\{z^n \bar{\partial}h_0 : n \ge 0\} \bigvee \bar{\partial}W_1^{q,0}(G). \tag{5.3}$$

Thus, to prove that $L_{\lambda}\mathcal{M}(E,I) \subset \mathcal{M}(E,I)$ for all $\lambda \in G$, we just need to show that for fixed $f \in \mathcal{M}(E,I)$ and $n \ge 0$, the function

$$H(\lambda) = \int_G L_{\lambda} f \bar{\partial}(z^n h_0)\,dA = 0 \quad \forall \lambda \in G.$$

(Notice that

$$\int_G L_{\lambda} f \bar{\partial}k\,dA = 0$$

for all $k \in W_1^{q,0}(G)$ by Lemma 3.2.) By [7], p. 30, H is analytic on G. From (3.5),

$$\begin{aligned} H(\lambda) &= \int_G \frac{f z^n \bar{\partial}h_0}{z-\lambda}\,dA - f(\lambda)\int_G \frac{\bar{\partial}(z^n h_0)}{z-\lambda}\,dA \\ &= -\pi C(f z^n \bar{\partial}h_0)(\lambda) + \pi f(\lambda)\lambda^n h_0(\lambda) \end{aligned}$$

and by construction, h_0 is zero in some annulus $G' = \{z : \rho' < |z| < \rho\}$. Thus

$$H(\lambda) = -\pi C(f z^n \bar{\partial}h_0)(\lambda) \quad \forall \lambda \in G'.$$

Since $\bar{\partial}h_0 \in L^{\infty}$, then $f\bar{\partial}h_0 \in L^p(G)$. In addition, $C(f z^n \bar{\partial}h_0)(\lambda) = 0$ for all $|\lambda| > \rho$ and so by (3.3), $C(f z^n \bar{\partial}h_0) \in W_1^p$. By Proposition 3.4,

$$H(\rho\zeta) = \lim_{r \to 1^-} H(r\rho\zeta)$$

exists almost everywhere and in $L^p(\rho\mathbb{T}, |d\zeta|)$ and so $H \in H^p(G')$. By Proposition 3.4 again,

$$H(\rho\zeta) = \lim_{r \to 1^+} H(r\rho\zeta) = 0$$

almost everywhere, since $H(\lambda) = C(z^n f \bar{\partial}h_0)(\lambda) = 0$ for $|\lambda| > \rho$. Thus H is an $H^p(G')$ function whose boundary values vanish a.e. on $\rho\mathbb{T}$, making $H \equiv 0$. □

Proof of Proposition 5.1: If $1 \in \mathcal{M}(E,I)$, then by the invariance of $\mathcal{M}(E,I)$, and the density of polynomials in $L^p_a(\rho\mathbb{D})$, $L^p_a(\rho\mathbb{D})|_G \subset \mathcal{M}(E,I)$. Thus if $f \in \mathcal{M}(E,I)$, then $f_+ \in \mathcal{M}(E,I)$ and hence $f_- \in \mathcal{M}(E,I)$.

To prove (1) we proceed as follows: By Lemma 5.2, $L_\lambda f_- \in \mathcal{M}(E,I)$ and so with h_0 chosen as is Lemma 5.2,

$$\int_G \frac{f_- - f_-(\lambda)}{z-\lambda} \overline{\partial} h_0 dA = 0 \quad \forall \lambda \in G.$$

From this one obtains the formula

$$f_-(\lambda) = \frac{C(f_-\overline{\partial}h_0)(\lambda)}{C(\overline{\partial}h_0)(\lambda)} = \frac{C(f_-\overline{\partial}h_0)(\lambda)}{h_0(\lambda)}, \quad \forall 1 < |\lambda| < \frac{1}{2}\rho, h_0(\lambda) \neq 0. \tag{5.4}$$

(Notice that h_0 is anti-analytic in this region and so $h_0(\lambda)$ is zero on only a countable set of points and so the above formula makes sense.)

As argued in the proof of Lemma 5.2, $C(f_-\overline{\partial}h_0) \in W^{p,0}_1(\rho\mathbb{D})$. By Proposition 3.4,

$$\sup_{1<r<\frac{1}{2}\rho} \int_{\mathbb{T}} |C(f_-\overline{\partial}h_0)(r\zeta)|^p |d\zeta| < \infty.$$

Hence from (5.4)

$$\sup_{1<r<\frac{1}{2}\rho} \int_{\mathbb{T}} |(f_-h_0)(r\zeta)|^p |d\zeta| < \infty. \tag{5.5}$$

In the region $1 < |z| < \frac{1}{2}\rho$, $h_0(z) = (I\mathcal{O})(1/\overline{z})$. Thus if $\Omega(z) = \overline{(I\mathcal{O})(1/\overline{z})}$ then $f_-\Omega$ is analytic on $\mathbb{D}_e$ and

$$\sup_{1<r<\frac{1}{2}\rho} \int_{\mathbb{T}} |(f_-\Omega)(r\zeta)|^p |d\zeta| = \sup_{1<r<\frac{1}{2}\rho} \int_{\mathbb{T}} |(f_-h_0)(r\zeta)|^p |d\zeta| < \infty,$$

by (5.5). Thus we have shown (1).

For the proof of (2) we define

$$\tilde{f}_-(\lambda) = \frac{C(f_-\overline{\partial}h_0)(\lambda)}{h_0(\lambda)}, \quad |\lambda| < 1. \tag{5.6}$$

The function $C(f_-\overline{\partial}h_0)|_{\mathbb{D}}$ is analytic on $\mathbb{D}$ and also belongs to D_p (since $C(f_-\overline{\partial}h_0) \in W^{p,0}_1(\rho\mathbb{D})$). Furthermore $h_0|_{\mathbb{D}} = I\mathcal{O} \in A^\infty$ and so $\tilde{f}_- \in N(\mathbb{D})$. By part (1), $f_- \in N(\mathbb{D}_e)$ and so the non-tangential limits of f_- and $\tilde{f}_-$ exits a.e. and by the choice of h_0 (being continuous on $\rho\mathbb{D}$), (5.4), (5.6), and Proposition 3.4,

$$\lim_{r\to 1^-} \tilde{f}_-(r\zeta) = \lim_{r \to 1^+} f_-(r\zeta) \quad a.e.$$

Thus $\tilde{f}_-$ is a pseudo-continuation of f_- and the proof of (2) is complete.

For the proof of (3), we recall from (5.2) that $h_0 = I\mathcal{O}$ on $\mathbb{D}$, where $\mathcal{O}$ is outer. Thus from (5.6), $I\tilde{f}_- \in N^+(\mathbb{D})$ and the proof is complete. □

Proof of Theorem 2.1: First notice that $\mathcal{M} = \mathcal{M}(E, I) \neq L^p_a(G)$ and so by (4.3), condition (2.1) holds. From Proposition 5.1, if $f \in \mathcal{M}(E, I)$, then conditions (1) and (2) hold. Now suppose $f \in L^p_a(G)$ satisfies (1) and (2). By (5.3) and the fact that

$$\int_G f_- \overline{\partial} k dA = 0 \ \ \forall k \in W^{q,0}_1(G),$$

to show $f_- \in \mathcal{M}(E, I)$ it suffices to show

$$\int_G f_- p \overline{\partial} h_0 dA = 0 \tag{5.7}$$

for all polynomials p.

To show (5.7), we proceed as follows: First notice that since $f_- \in N(\mathbb{D}_e)$ and h_0 is continuous, then

$$\lim_{r \to 1^+} f_-(r\zeta) h_0(r\zeta) = f(\zeta) h_0(\zeta) \ \ a.e.$$

In addition, since $f_- \overline{h_0} = f_- \Omega$ is H^1 near $\mathbb{T}$, then by [6], Theorem 2.6, p. 21,

$$\begin{aligned} \lim_{r \to 1^+} \int_{\mathbb{T}} |f_-(r\zeta) h_0(r\zeta)||d\zeta| &= \lim_{r \to 1^+} \int_{\mathbb{T}} |f_-(r\zeta)\Omega(r\zeta)||d\zeta| \\ &= \int_{\mathbb{T}} |f_-(\zeta)\Omega(\zeta)||d\zeta| \\ &= \int_{\mathbb{T}} |f_-(\zeta) h_0(\zeta)||d\zeta|. \end{aligned}$$

Applying [6], Lemma 1, p. 21, we get

$$\lim_{r \to 1^+} \int_{\mathbb{T}} |f_-(r\zeta) h_0(r\zeta) - f_-(\zeta) h(\zeta)||d\zeta| = 0. \tag{5.8}$$

Then we can say

$$\begin{aligned} \int_G f_- p \overline{\partial} h_0 dA &= \lim_{r \to 1^+} \int_{G \cap \{|z| > r\}} \overline{\partial}(f_- p h_0) dA \\ &= -\frac{1}{2i} \lim_{r \to 1^+} \int_{\mathbb{T}} f_-(r\zeta) p(r\zeta) h_0(r\zeta) d\zeta, \quad \text{by Green's theorem } (h_0|_{\rho\mathbb{T}} = 0) \\ &= -\frac{1}{2i} \int_{\mathbb{T}} (f_- h_0)(\zeta) p(\zeta) d\zeta, \quad \text{by (5.8)} \\ &= -\frac{1}{2i} \int_{\mathbb{T}} (I\mathcal{O} f_-)(\zeta) p(\zeta) d\zeta \\ &= 0, \end{aligned}$$

since $If_- \in N^+(\mathbb{D})$ and $I\mathcal{O}f_- \in L^1(\mathbb{T})$, and so $I\mathcal{O}f_- \in H^1(\mathbb{T})$, [6], p. 28, Theorem 2.11. Thus we have shown (5.7) and the proof is done. □.

Proof of Corollary 2.2: From (5.6), f_- and $\tilde{f}_-$ are analytic on $\mathbb{C}_\infty \backslash \mathbb{D}$ and $\mathbb{D} \backslash \text{spec}(I)$ respectively. Since $\tilde{f}_-$ is a pseudo-continuation of f_-, it suffices to show that $\tilde{f}_-$ is an analytic continuation of f_- across each arc of $\mathbb{T}$ disjoint from $E \cup \text{spec}(I)$. Let J be such an arc. Since $(I\mathcal{O})^{-1}(0) = E \cup \text{spec}(I)$, then there is a neighborhood U of J such that h_0 is bounded away from zero in U.

Note that

$$f_-(\lambda) = \frac{C(f\overline{\partial} h_0)(\lambda)}{h_0(\lambda)}, \quad |\lambda| \sim 1$$

and thus using (5.5) and the fact that h_0 is bounded away from zero in U, we get that $f_- \in H^1(U \cap \{|z| > 1\})$. A similar argument says that $\tilde{f}_- \in H^1(U \cap \{|z| < 1\})$. In addition, $f_- = \tilde{f}_-$ a.e. on $U \cap \mathbb{T}$, and thus by [20], p. 41, $\tilde{f}_-$ is an analytic continuation of f_- across J. □

Proof of Corollary 2.3: If $f \in \mathcal{M}(E, I)$, then by Theorem 2.1 and Corollary 2.2, conditions (1) - (3) hold. For the other direction, we suppose $f \in L^p_a(G)$ with (1) - (3) holding. Then by (4.3) we just need to show that

$$\int_G f_- \overline{\partial} h dA = 0 \tag{5.9}$$

whenever $h \in W_1^{q,0}(\rho\mathbb{D})$ with $h|_{\mathbb{D}} \in \mathcal{F}(E, I)$. It is known (see the remark following this proof) that the functions $h \in \mathcal{F}(E, I)$ for which

$$|h(z)| \leq C\text{dist}^4(z, E \cup \text{spec}(I)), \quad |z| < 1 \tag{5.10}$$

are dense in $\mathcal{F}(E, I)$. Thus we just need to verify (5.9) for $h|_{\mathbb{D}}$ satisfying (5.10). Furthermore, if $h_1 \in W_1^{q,0}(\rho\mathbb{D})$ with $h_1 = h$ on $\mathbb{D}$, then $h_1 - h \in W_1^{q,0}(G)$ (since it is zero on $\mathbb{D}$, Proposition 3.1) and so

$$\int_G f_- \overline{\partial}(h_1 - h) dA = 0.$$

Hence we can assume, for $|z| > 1$ and near $\mathbb{T}$, that $h(z) = h(\frac{1}{\bar{z}})$ and so

$$|h(z)| \leq C\text{dist}^4(z, E \cup \text{spec}(I)), \quad |z| \sim 1. \tag{5.11}$$

Moreover if f satisfies (1) - (3), then

$$|f_-(z)| \leq C\text{dist}^{-4}(z, E \cup \text{spec}(I)), \quad 1 < |z| < 2. \tag{5.12}$$

We defer the proof of this fact to the next lemma and note that here is where the hypothesis of a finite Blaschke product is being used. Thus for h satisfying (5.11),

$$\begin{aligned}
\int_G f_- \overline{\partial} h dA &= \lim_{r \to 1^+} \int_{G \cap \{|z| > r\}} \overline{\partial}(f_- h) dA \\
&= -\frac{1}{2i} \lim_{r \to 1^+} \int_{\mathbb{T}} f_-(r\zeta) h(r\zeta) d\zeta, \quad \text{Green's theorem } (h|_{\rho\mathbb{T}} = 0) \\
&= -\frac{1}{2i} \int_{\mathbb{T}} f_-(\zeta) h(\zeta) d\zeta.
\end{aligned}$$

This last equality is justified since, by (5.10) and (5.12), $f_- h$ is uniformly bounded in $1 < |z| < 2$. But $If_- \in N^+(\mathbb{D})$, $f_- h \in L^\infty(\mathbb{T})$, and $h/I \in H^\infty(\mathbb{D})$. Thus $f_- h \in H^\infty(\mathbb{D})$ and so the above integral is zero. □

Remark: The fact that

$$\mathrm{clos}_{D_q}\{h \in \mathcal{F}(E,I) : |h(z)| \le C\mathrm{dist}^4(z, E \cup \mathrm{spec}(I))\} = \mathcal{F}(E,I) \tag{5.13}$$

was proved by Shirokov in [21], though not explicitly stated there. Shirokov proves various technical lemmas and estimates necessary to prove the analog of a result of Korenblum [11], Theorem 4.1, for the Dirichlet space (Besov space). The analog of [11], Theorem 4.1 is precisely (5.13), and is the key step in characterizing the ideals of D_q.

We end this section with the proof of the estimate (5.12) but first we make a few preliminary comments. If $z \in G$ and $r > 0$ with $B(z,r) = \{|w - z| < r\} \subset G$, then by the mean value theorem for harmonic functions,

$$f_-(z) = \frac{1}{\pi r^2}\int_{B(z,r)} f(w)dA(w).$$

Thus

$$|f_-(z)| \le \frac{1}{\pi r^2}\int_G |f(w)|dA(w).$$

An application of Holder's inequality yields

$$|f_-(z)| \le C_f(|z| - 1)^{-2}, \quad |z| > 1. \tag{5.14}$$

Let $\mathcal{H}(\mathbb{D})$ denote the analytic functions on $\mathbb{D}$. If $f \in N(\mathbb{D}) \cap \mathcal{H}(\mathbb{D})$, then $\log|f(z)|$ has least harmonic majorant the Poisson integral of a finite measure μ on $\mathbb{T}$ [8], p. 69. That is to say

$$\log|f(z)| \le \int_{\mathbb{T}} P_z(\theta)d\mu(\theta),$$

where $P_z(\theta)$ is the usual Poisson kernel

$$P_z(\theta) = \Re\frac{e^{i\theta} + z}{e^{i\theta} - z}.$$

From this one obtains

$$|f(z)| \le \exp\{\frac{C_f}{1 - |z|}\}. \tag{5.15}$$

Lemma 5.3. *Let $I = BS_\mu$ be inner with B a finite Blaschke product and $f \in L^p_a(G)$ satisfy the following conditions:*

(1) *f_- has an analytic continuation $G(z)$ to $\mathbb{C}_\infty \backslash (E \cup spec(I))$.*
(2) *$IG \in N^+(\mathbb{D})$.*

Then the following estimate holds

$$|G(z)| \le C\, dist^{-4}(z, E \cup spec(I)), \quad 1 \le |z| \le 2. \tag{5.16}$$

Proof. The idea of the proof comes from [11]. (We remark to the reader that in [11], the notation for $N^+(\mathbb{D})$ is slightly different from ours.) Without loss of generality we assume that the zeros of B (which are finite) are contained in the disk $\{|z| < 1/2\}$. Use the fact that $BG \in N(\mathbb{D}) \cap \mathcal{H}(\mathbb{D})$ (i.e. condition (2)) along with (5.15) to get

$$|BG(z)| \le \exp\{\frac{C}{1-|z|}\}, \quad |z| < 1.$$

Substituting a larger constant C if necessary we get

$$|G(z)| \le \exp\{\frac{C_G}{1-|z|}\}, \quad \frac{1}{2} < |z| < 1. \tag{5.17}$$

Let

$$\gamma = \{\zeta \in \mathbb{T}, a < arg(\zeta) < b\}$$

be one of the complimentary arcs of $E \cup \mathrm{spec}(I)$. Without loss of generality we will assume that $|\gamma| < 1$. Letting $(a', b') \subset [a, b]$ we define

$$\psi_1(z) = \Big[\, \frac{(z - e^{ia'})(z - e^{ib'})}{z - (1 + |\gamma|)e^{ia'}} \,\Big]^4$$

$$\psi_2(z) = \exp\Big\{\, C_G\, \Big(\, \frac{e^{ia'}}{z - e^{ia'}} + \frac{e^{ib'}}{z - e^{ib'}} \,\Big)\Big\}.$$

Define

$$\tilde{G}(z) = \psi_1(z)\psi_2(z)G(z)$$

and note by condition (1) that $\tilde{G}$ is analytic in $Q_{a',b'}$, where $Q_{a',b'}$ is the 5-sided region bounded by $\{\frac{1}{2}\zeta : a' \le \arg(\zeta) \le b'\}$, $\{re^{ia'} : \frac{1}{2} \le r \le 1\}$, $\{re^{ib'} : \frac{1}{2} \le r \le 1\}$, and the tangents to $\mathbb{T}$ at the points $e^{ia'}$ and $e^{ib'}$. Now use (5.14), (5.17) along with the maximum modulus principle to get

$$|\tilde{G}(z)| \le C, \quad \forall z \in Q_{a',b'}, \tag{5.18}$$

Moreover

$$\psi_1(\zeta)\psi_2(\zeta) \simeq \max\{|\zeta - e^{ia'}|, |\zeta - e^{ib'}|\}^4, \quad \zeta \in \mathbb{T}, \quad a' \le \arg(\zeta) \le b'.$$

Now using (5.18) and passing to a limit as $(b - a) - (b' - a') \to 0$ we have

$$|G(\zeta)| \le C\mathrm{dist}^{-4}(\zeta, E \cup \mathrm{spec}(I)), \quad \zeta \in \gamma, \tag{5.19}$$

where the constant C does not depend on the arc γ. Thus we have proved (5.16) for $\zeta \in \mathbb{T}$.

Now look at the region

$$Q_\gamma = \{z : 1 \le |z| \le 2, a \le \arg(z) \le b\}$$

and the function $\psi_3(z)G(z)$, where

$$\psi_3(z) = \Big[\, \frac{(z - e^{ia})(z - e^{ib})}{z - (1 - |\gamma|)e^{ia}} \,\Big]^4.$$

Again, using the maximum modulus principle along with (5.14) and (5.19), we obtain

$$|\psi_3(z)G(z)| \le C, \quad z \in Q_\gamma,$$

where C does not depend on the arc γ. Since

$$|\psi_3(z)| \simeq \operatorname{dist}^4(z, E \cup \operatorname{spec}(I)), \quad z \in Q_\gamma,$$

we have shown (5.16) for $1 \le |z| \le 2$. □

6. Application: The adjoint of the shift on the Dirichlet space

For $q > 2$, the invariant subspaces of the Dirichlet spaces D_q are known to be $\mathcal{F}(E, I)$ for some $E \subset \mathbb{T}$ and some inner function I. We will now characterize the invariant subspaces of M_z', the adjoint of M_z on D_q, or equivalently the annihilator of $\mathcal{F}(E, I)$. We point out that in order to characterize the invariant subspaces of D_q, $q > 2$, Shirokov [21] obtained some information on the annihilators of invariant subspaces, such as analytic continuation properties. We will also obtain these same results along with a complete characterization of these annihilators. We begin with a known fact for which we include a brief sketch of the proof for completeness.

Proposition 6.1. *For $1 < p < \infty$, the dual of D_p is D_q via the bi-linear pairing*

$$(g, h) \stackrel{def}{=} \int_{\mathbb{T}} g(\zeta)h(\overline{\zeta})\frac{|d\zeta|}{2\pi} + \int_{\mathbb{D}} g'(z)h'(\overline{z})\frac{dA(z)}{\pi} \quad g \in D_p, h \in D_q. \tag{6.1}$$

Proof. It is well known, for example [25], p. 95, that the dual of D_p is D_q via the pairing

$$\int_D f'(z)g'(\overline{z})\frac{dA(z)}{\pi}, \quad f \in D_p, g \in D_q.$$

Next we notice that the operator

$$(Hf)(z) = \frac{1}{z}\int_0^z wf'(w)dw$$

is a well defined continuous operator from D_q to D_q. (This is because

$$H = M_{1/z}FM_z\frac{d}{dz},$$

where $(Ff)(z)$ is the anti-derivative of f which vanishes at $z = 0$. The operator $\frac{d}{dz}$ is continuous from D_q to $L^q_a(\mathbb{D})$, the operator M_z is continuous on $L^q_a(\mathbb{D})$, the operator F is continuous from $L^q_a(\mathbb{D})$ to zD_q, and $M_{1/z}$ is continuous from zD_q to D_q.)

Next notice that

$$Hz^n = \frac{n}{n+1}z^n, \quad n = 0, 1, 2, \cdots$$

and so

$$\int_{\mathbb{D}} f'(z)g'(\overline{z})\frac{dA(z)}{\pi} = \int_{\mathbb{T}} f(\zeta)(Hg)(\overline{\zeta})\frac{|d\zeta|}{2\pi} + \int_{\mathbb{D}} f'(z)(Hg)'(\overline{z})\frac{dA(z)}{\pi}$$

for all polynomials f and g. Now use the density of polynomials in D_p and D_q to obtain the final result. □

A simple calculation using power series yields

$$(g,h)=\lim_{r\to 1^+}\int_{\mathbb{T}}(zg)'(\frac{1}{r\zeta})h(\frac{1}{r}\zeta)\frac{|d\zeta|}{2\pi}. \tag{6.2}$$

If M_z' denotes the adjoint of M_z (multiplication by z) on D_q, then $M_z' : D_p \to D_p$ with

$$(M_z'g,h)=(g,zh),\quad g\in D_p, h\in D_q.$$

From this one sees that the invariant subspaces of M_z' are precisely the annihilators of the $\mathcal{F}(E,I)$ spaces.

Theorem 6.2. *If $\mathcal{F}(E,I)$ is a non-trivial ideal of D_q, $q>2$, then $g\in D_p$ annihilates $\mathcal{F}(E,I)$ (i.e. $(g,h)=0$ for all $h\in\mathcal{F}(E,I)$) if and only if the following two conditions are satisfied:*

(1) *$(zg)'(1/z)\Omega\in H^1(\mathbb{D}_e)$, where $\Omega(z)=\overline{(IO)(1/\overline{z})}$ and IO is the Korenblum function.*
(2) *$(zg)'(1/z)$ has a pseudo continuation $G(z)$ to $\mathbb{D}$ with $IG\in zN^+(\mathbb{D})$.*

Moreover, $(zg)'(1/z)$ has an analytic continuation to $\mathbb{C}_\infty\setminus(E\cup spec(I))$. In addition, If $I=BS_\mu$ and B is a finite Blaschke product, then $g\in D_p$ annihilates $\mathcal{F}(E,I)$ if and only if the following conditions are satisfied:

(1) *$(zg)'(1/z)\in N(\mathbb{D}_e)$*
(2) *$(zg)'(1/z)$ has an analytic continuation G to $\mathbb{C}_\infty\setminus(E\cup spec(I))$.*
(3) *$IG\in zN^+(\mathbb{D})$.*

Proof. We will prove the first part of the theorem and notice that from this the other two parts will follow immediately. Before we get to the main body of the proof, we first make a preliminary observation. Given $h\in\mathcal{F}(E,I)$, let $h_1\in W^{q,0}_{1,A}(\rho\mathbb{D})$ with $h_1|_{\mathbb{D}}=h$ and $h_1(z)=h(\frac{1}{\bar{z}})$ for $|z|>1$ and near $\mathbb{T}$. If $g\in D_p$, then an easy calculation shows that $(zg)'(1/z)\in L^p_a(G)$ and

$$\begin{aligned}(g,zh) &= \lim_{r\to1^+}\int_{\mathbb{T}}(zg)'(\frac{1}{r\zeta})\frac{1}{r}\zeta h(\frac{1}{r}\zeta)\frac{|d\zeta|}{2\pi}\\ &= \frac{i}{2\pi}\lim_{r\to1^+}\int_{r\mathbb{T}}(zg)'(\frac{1}{z})h_1(z)dz\\ &= \frac{1}{\pi}\lim_{r\to1^+}\int_{G\cap\{|z|>r\}}\overline{\partial}((zg)'(\frac{1}{z})h_1)dA, \quad \text{Green's theorem}\\ &= \frac{1}{\pi}\int_G(zg)'(\frac{1}{z})\overline{\partial}h_1dA.\end{aligned}$$

If h_2 is any function in $W^{q,0}_{1,A}(\rho\mathbb{D})$ with $h_2|_{\mathbb{D}}=h$, then $h_1-h_2=0$ off G and so by Proposition 3.1, $h_1-h_2\in W^{q,0}_1(G)$. By Lemma 3.2, $\overline{\partial}(h_1-h_2)\in L^p_a(G)^{\perp}$ and so

$$\int_G(zg)'(\frac{1}{z})\overline{\partial}h_1dA=\int_G(zg)'(\frac{1}{z})\overline{\partial}h_2dA.$$

We conclude from this that if $g \in D_p$ and $h \in D_q$ with $\tilde{h} \in W^{q,0}_{1,A}(\rho\mathbb{D})$ and $\tilde{h}|_{\mathbb{D}} = h$, then

$$(g, zh) = \frac{1}{\pi}\int_G (zg)'(\frac{1}{z})\bar{\partial}\tilde{h}\,dA. \tag{6.3}$$

From this one has that if $(g,h) = 0$ for all $h \in \mathcal{F}(E,I)$, then $(g, zh) = 0$ for all $h \in \mathcal{F}(E,I)$ and so by (6.3) and (4.3), $(zg)'(1/z) \in \mathcal{M}(E,I)$. By Theorem 2.1 $(zg)'(1/z)\Omega \in H^1(\mathbb{D}_e)$ and $(zg)'(1/z)$ has a pseudo-continuation G to $\mathbb{D}$ with $IG \in N^+(\mathbb{D})$. Moreover, since $(zg)'(1/z)\Omega \in H^1(\mathbb{D}_e)$, then, as argued in the proof of Theorem 2.1,

$$\begin{aligned} 0 = (g, I\mathcal{O}) &= \lim_{r\to 1^+}\int_{\mathbb{T}}(zg)'(\frac{1}{r\zeta})(I\mathcal{O})(\frac{1}{r}\zeta)\frac{|d\zeta|}{2\pi} \\ &= \int_{\mathbb{T}} G(\zeta)(I\mathcal{O})(\zeta)\frac{|d\zeta|}{2\pi} \\ &= G(0)(I\mathcal{O})(0), \end{aligned}$$

hence $IG \in zN^+(\mathbb{D})$ and conditions (1) and (2) hold.

Conversely suppose conditions (1) and (2) hold. Then for all $n \geq 0$, $(z^n GI\mathcal{O})(0) = 0$ and by the calculation above (using again the fact that $(zg)'(1/z)\Omega \in H^1(\mathbb{D}_e)$),

$$0 = (z^n GI\mathcal{O})(0) = (g, z^n I\mathcal{O}).$$

From this one concludes that g annihilates $\mathcal{F}(E,I)$. □

Remarks:

(1) By a technique of Korenblum [11], Shirokov [21] proved that if $g \in D_p$ annihilates $z\mathcal{F}(E,I)$, then the Borel transform

$$(g, \frac{z}{z-\zeta}), \quad |z| > 1$$

has an analytic continuation to $\mathbb{C}_\infty \backslash (E \cup \text{spec}(I))$. (He actually makes this observation in a more general setting than the Dirichlet space.) A calculation using (6.2) yields

$$(g, \frac{z}{z-\zeta}) = (zg)'(\frac{1}{z})$$

and thus the analytic continuation property was really first observed by Shirokov, though his proof is much different from ours.

(2) For the classical Dirichlet space D_2, though the invariant subspaces are not known, it has been observed by Richter and Sundberg [16] that if $g \in D_2$ annihilates a non-trivial invariant subspace $\mathcal{F} \subset D_2$, then $(zg)'(1/z) \in N(\mathbb{D}_e)$ and has a pseudo-continuation to $N(\mathbb{D})$, with an analytic continuation to $\mathbb{C}_\infty \backslash \underline{Z}(\mathcal{F})$. Here $\underline{Z}(\mathcal{F})$ is the so-called "lim-inf" zero set of $\mathcal{F}$, see [16] for a definition. We note however that unlike the D_q $(q > 2)$ Dirichlet space where $Z(\mathcal{F}(E,I)) \cap \mathbb{T}$ is a Carleson set, the lim-inf zero set $Z(\mathcal{F})$ for an invariant subspace of D_2 can have the property $Z(\mathcal{F}) \cap \mathbb{T} = \mathbb{T}$, see [16], Theorem 4.3.

7. Application: The backward Bergman shift

In this section, we will characterize the invariant subspaces of the backward Bergman shift

$$L : L^p_a(\mathbb{D}) \to L^p_a(\mathbb{D}), \quad Lf = \frac{f - f(0)}{z},$$

for $1 < p < 2$. We first begin with a proposition which relates L with M_z', the adjoint of the Dirichlet shift.

Proposition 7.1. *The operator*

$$U : D_p \to L^p_a(\mathbb{D}), \quad (Ug)(z) = (zg)'(z)$$

is continuous and invertible with $LU = UM_z'$.

Proof. To prove the continuity of U we first remark that for $g \in D_p$,

$$\|g\|^p_{L^p} \leq C(|g(0)| + \|g'\|^p_{L^p}),$$

see [25], p. 58. Thus

$$\|Ug\|_{L^p} = \|(zg)'\|_{L^p} \leq C\|g\|_{D_p}.$$

The Dirichlet shift is bounded below and so

$$\|Ug\|_{L^p} = \|(zg)'\|_{L^p} = \|zg\|_{D_p} \geq C\|g\|_{D_p},$$

which makes U bounded below and thus have closed range. Since

$$Uz^n = (z+1)z^n, \quad n = 0, 1, \cdots \tag{7.1}$$

then U has dense range and so U is invertible.

A calculation shows that

$$Lz^n = \begin{cases} 0 & n = 0, \\ z^{n-1} & n > 0 \end{cases} \qquad M_z' z^n = \begin{cases} 0 & n = 0, \\ \frac{n+1}{n} z^{n-1}, & n > 0 \end{cases}$$

Now combine this with (7.1) along with the density of polynomials in both D_p and $L^p_a(\mathbb{D})$ to get $LU = UM_z'$. $\square$

To prove Theorem 2.4 we apply Proposition 7.1 to Theorem 6.2.

We now turn to the question of cyclic vectors for the backward Bergman shift. It is known [20] that a vector $f \in H^p(\mathbb{D})$ is a *cyclic vector* for the backward shift L on the Hardy space, i.e.

$$\text{span}_{H^p}\{f, Lf, L^2 f, \cdots\} = H^p(\mathbb{D})$$

if and only if $f(\frac{1}{z})$ does not have a pseudo-continuation to $\mathbb{D}$. We will now determine the cyclic vectors f for the backward Bergman shift, i.e.

$$\text{span}_{L^p_a}\{f, Lf, L^2 f, \cdots\} = L^p_a(\mathbb{D})$$

for $1 < p < 2$.

Theorem 7.2. *For $1 < p < 2$, a vector $f \in L^p_a(\mathbb{D})$ is not cyclic for L if and only if there is a closed set $E \subset \mathbb{T}$ and an inner function I satisfying (2.1) such that*

(1) $f(\frac{1}{z})\Omega \in H^1(\mathbb{D}_e)$.

(2) *$f(\frac{1}{z})$ has a pseudo-continuation to $\mathbb{D}$ with $If(\frac{1}{z}) \in zN^+(\mathbb{D})$.*

Proof. If f is not L-cyclic, then

$$[f]_L = \text{span}_{L^p_a}\{f, Lf, L^2 f, \cdots\}$$

is a proper (closed) L-invariant subspace of $L^p_a(\mathbb{D})$. An application of Theorem 2.4 gives the closed set E and the inner function I satisfying (2.1) with conditions (1) and (2) holding.

Conversely, if there is a E and I satisfying (2.1) then by Theorem 2.4, the space $\mathcal{K}(E, I)$ is a proper (closed) L-invariant subspace of $L^p_a(\mathbb{D})$. If $f \in L^p_a(\mathbb{D})$ satisfies (1) and (2), then $f \in \mathcal{K}(E, I)$ and

$$[f]_L \subset \mathcal{K}(E, I) \neq L^p_a(\mathbb{D}).$$

Thus f is not L-cyclic. $\square$

Remarks:

(1) There are a variety of examples of L-cyclic vectors for the backward Bergman shift on $L^p_a(\mathbb{D})$. For example, any $f \in H^p(\mathbb{D})$ which is L-cyclic for $H^p(\mathbb{D})$ is also L-cyclic for $L^p_a(\mathbb{D})$. We refer the reader to [20] for many examples of cyclic vectors for $H^p(\mathbb{D})$. One can also create examples of L-cylic vectors for $L^p_a(\mathbb{D})$ just by taking $f \in L^p_a(\mathbb{D})\backslash N(\mathbb{D})$, e.g. take $f \in L^p_a(\mathbb{D})$ with $f^{-1}(0)$ not a Blaschke sequence (note that such vectors exist).

(2) For $H^p(\mathbb{D})$, a vector f is L-cyclic for the backward Hardy space shift if and only if $f(\frac{1}{z})$ does not have a pseudo-continuation to $\mathbb{D}$. For $L^p_a(\mathbb{D})$, $1 < p < 2$, the L-cyclic vectors cannot be characterized by pseudo-continuations alone as can be seen by the following example: Every inner function ϕ is not cyclic for $H^p(\mathbb{D})$ since

$$\tilde{\phi}(z) = \overline{\frac{1}{\phi}}(\overline{z}), \quad |z| < 1$$

is a pseudo-continuation of $\phi(\frac{1}{z})$ to $\mathbb{D}$. One can see from Theorem 7.2 that ϕ is L-cyclic for $L^p_a(\mathbb{D})$, $1 < p < 2$, if and only if

$$\int_{\mathbb{T}} \log \text{dist}(\zeta, \text{spec}(\phi))|d\zeta| = -\infty.$$

Chosing a inner function ϕ which satisfies the above condition (e.g. $\phi = S_\mu$, where μ is a singular measure whose support is not a Carleson set, or $\phi = B$, where B is a Blaschke product whose zeros accumulate on a subset of $\mathbb{T}$ which is not a Carleson set) will give an example of a function for which $\phi(\frac{1}{z})$ has a pseudo-continuation to $\mathbb{D}$, and hence is not L-cyclic for $H^p(\mathbb{D})$, but is L-cyclic for $L^p_a(\mathbb{D})$, $1 < p < 2$.

(3) For $p = 2$ it is known that if $g \in D_2$ annihilates an non-trivial z-invariant subspace, then $(zg)'(\frac{1}{z})$ has a pseudo-continuation to $\mathbb{D}$. The backward Bergman shift L on $L^2_a(\mathbb{D})$ is similar to M'_z (on D_2) via U as before. Thus if $f \in L^2_a(\mathbb{D})$ is not L-cylic, then $f(\frac{1}{z})$ has a pseudo-continuation to $\mathbb{D}$. Since we do not have a complete characterization of the z-invariant (nor the M'_z-invariant) subspaces of D_2, as one does for D_q, $q > 2$, we do not have a complete description of the L-cyclic vectors for $L^2_a(\mathbb{D})$.

Finally, we mention that our results can be used to characterize the so-called nearly invariant subspaces of the Bergman space of the exterior disk. For $1 < p < \infty$, let $L^p_a(\mathbb{D}_e)$ be the space of analytic functions on $\mathbb{D}_e$ with

$$\int_{\mathbb{D}_e} |f(z)|^p \frac{1}{|z|^4} dA(z) < \infty.$$

Define the operator

$$L_\infty : L^p_a(\mathbb{D}_e) \to L^p_a(\mathbb{D}_e), \quad L_\infty f = z(f - f(\infty)).$$

A subspace $\mathcal{N} \subset L^p_a(\mathbb{D}_e)$ is said to be *nearly invariant* if $L_\infty \mathcal{N} \subset \mathcal{N}$. The term 'nearly invariant' was coined by D. Sarason [19] in his investigations of $H^p(\mathbb{D}_e)$. We can now give a characterization of the nearly invariant subspaces of $L^p_a(\mathbb{D}_e)$ for $1 < p < 2$.

Define the operator

$$U_\infty : D_p \to L^p_a(\mathbb{D}_e) \quad (Ug)(z) = (zg)'(\frac{1}{z})$$

and notice, from a similar type of calculation as in Section 6, that U_∞ is continuous and invertible with

$$L_\infty U_\infty = U_\infty M'_z.$$

This allows us to apply Theorem 6.2 to characterize the nearly invariant subspaces as follows:

Theorem 7.3. *Let $1 < p < 2$ and $\mathcal{N} \neq L^p_a(\mathbb{D})$ be nearly invariant. Then there is a closed set $E \subset \mathbb{T}$ and an inner function I satisfying (2.1) with $\mathcal{N} = \mathcal{N}(E, I)$, where $\mathcal{N}(E, I)$ is the space of functions $f \in L^p_a(\mathbb{D}_e)$ which satisfy*

(1) $f\Omega \in H^1(\mathbb{D}_e)$

(2) *f has a pseudo-continuation to $\mathbb{D}$ with $If \in zN^+(\mathbb{D})$.*

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