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Thomas Faulkner

Max Tegmark

Emory F. Bunn
University of Richmond, ebunn@richmond.edu

Yi Mao

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Constraining $f(R)$ gravity as a scalar-tensor theory

Thomas Faulkner,[1] Max Tegmark,[1,2] Emory F. Bunn,[3,2] and Yi Mao[1]
[1]*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[2]*MIT Kavli Institute for Astrophysics and Space Research, Cambridge, Massachusetts 02139, USA*
[3]*Physics Department, University of Richmond, Richmond, Virginia 23173, USA*
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We search for viable $f(R)$ theories of gravity, making use of the equivalence between such theories and scalar-tensor gravity. We find that models can be made consistent with solar system constraints either by giving the scalar a high mass or by exploiting the so-called chameleon effect. However, in both cases, it appears likely that any late-time cosmic acceleration will be observationally indistinguishable from acceleration caused by a cosmological constant. We also explore further observational constraints from, e.g., big bang nucleosynthesis and inflation.

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I. INTRODUCTION

Although the emerging cosmological standard model fits measurements spectacularly well (see [1,2] for recent reviews), it raises three pressing questions: what is the physics of the postulated dark matter, dark energy, and inflation energy? The need to postulate the existence of as many as three new substances to fit the data has caused unease among some cosmologists [3–6] and prompted concern that these complicated dark matter flavors constitute a modern form of epicycles. Our only knowledge about these purported substances comes from their gravitational effects. There have therefore been numerous suggestions that the apparent complications can be eliminated by modifying the laws of gravity to remove the need for dark matter [7,8], dark energy [9–11], and inflation [12], and perhaps even all three together [13]. Since attempts to explain away dark matter with modified gravity have been severely challenged by recent observations, notably of the so-called bullet cluster [14], we will focus on dark energy (hereinafter "DE") and inflation.

There is also a second motivation for exploring alternative gravity theories: observational constraints on parametrized departures from general relativity (GR) have provided increasingly precise tests of GR and elevated confidence in its validity [15,16].

A. $f(R)$ gravity

An extensively studied generalization of general relativity involves modifying the Einstein-Hilbert Lagrangian in the simplest possible way, replacing $R - 2\Lambda$ by a more general function $f(R)$ [11,20–35]. The equations of motion derived from this Lagrangian differ from Einstein's field equations when $f(R)$ is nonlinear, but the theory retains the elegant property of general coordinate invariance. In such a theory, the acceleration of our universe may be explained if $f(R)$ departs from linearity at small R, corresponding to late times in cosmological evolution. In this case it may be possible to avoid invoking a cosmological constant to explain cosmic acceleration, although one then replaces the problem of a small cosmological constant with the problem of no cosmological constant. In such models, the effective DE is dynamic, i.e., it is not equivalent to a cosmological constant, leading to potentially interesting observational signatures. We refer to these models as $f(R)$-DE theories.

In addition to potentially explaining late-time acceleration, $f(R)$ theories may be relevant to early-universe physics, particularly if $f(R)$ is nonlinear at large R [12,36]. More generally, it is of interest to study $f(R)$ theories because they are arguably the simplest setting in which one can attack the general question of which modified theories of gravity are allowed. By examining $f(R)$ theories, a broad class of theories containing GR as a special case, we continue the program of testing GR as best we can.

Of course, $f(R)$ theories are only a subset of theories based on modifications of the Einstein-Hilbert Lagrangian. For the general case where f depends on the full Riemann tensor $R^{\mu}_{\nu\alpha\beta}$ rather than merely on its contraction into the Ricci scalar R, this program is more complicated; a subset of these theories which are ghost free can be written as $f(R, G)$, where $G = R^{\beta}{}_{\mu\nu\alpha}R_{\beta}{}^{\mu\nu\alpha} - 4R^{\mu\nu}R_{\mu\nu} + R^2$ is the Gauss-Bonnet scalar in 4 dimensions [17]. These theories lack a simple description in terms of canonical fields; there is no so-called Einstein frame. Progress has nevertheless been made along these lines, and such Lagrangians may have more relevance to DE [17–19] than ones independent of G. In this paper we will not consider these Gauss-Bonnet theories further.

B. The equivalence with scalar-tensor gravity

The modified Einstein field equations (and so the new Friedmann equation) resulting from a nonlinear $f(R)$ in the action can be seen simply as the addition of a new scalar degree of freedom. In particular, it is well known that these theories are exactly equivalent to a scalar-tensor theory [37,38]. It is therefore no surprise that for $f(R)$-DE theories, it is this scalar which drives the DE. Before reviewing the mathematics of this equivalence in full detail in Sec. II, we will discuss some important qualitative features below.

One can discuss the theory in terms of the original metric $g_{\mu\nu}$, in which case the degrees of freedom are not manifest. Alternatively, by a conformal relabeling, one can reveal the theory to be regular gravity $\tilde{g}_{\mu\nu}$, plus scalar field ϕ. The former viewpoint is referred to as the Jordan frame (JF) and the latter as the Einstein frame (EF). Here ϕ has the peculiar feature that in the JF, it exactly determines the Ricci scalar R and vice versa. So in the JF, the Ricci scalar can in a sense be considered a noncanonical yet dynamical scalar field. This feature is absent in normal general relativity, where $R = -T/M_{\rm pl}^2 + 4\Lambda$ is algebraically fixed by the trace of the stress energy tensor T. Working in either frame is satisfactory as long as one is careful about what quantities are actually measurable, but we will find that the EF is much more useful for most of our calculations.

The coupling of the scalar field to matter is fixed in $f(R)$ gravity, and is essentially of the same strength as the coupling of the graviton to matter, except for the important case of massless conformally invariant fields, which do not couple to ϕ at all. The dynamics of the theory are completely specified by the potential $V(\phi)$ for the scalar field in the EF, which is uniquely determined by the functional form of $f(R)$.

After this lightning introduction to $f(R)$ theory we are ready to summarize our main motivations for studying $f(R)$ theories:

(i) There is recent renewed interest in this class of theories due to their possible relevance to DE.

(ii) These theories may have an interesting explanation in terms of a more complete theory of gravity.

(iii) Although there is an exact equivalence between $f(R)$ theories and a class of scalar-tensor theories, $f(R)$ theories may provide a new perspective on scalar-tensor theories. For example, a simple $f(R)$ may generate a complicated nontrivial scalar potential $V(\phi)$ that you would not have thought of using if just studying scalar-tensor theories.

(iv) Exploring modified or alternative theories is a way to test general relativity.

C. The $R - \mu^4/R$ example

Such a scalar field is not without observational consequence for solar system tests of gravity, especially for $f(R)$-DE models. For *any* scalar field driving DE, we can come to the following conclusions: First, the field value ϕ must vary on a time scale of order the Hubble time H_0^{-1}, if the DE is distinguishable from a cosmological constant (for a longer time scale, the DE looks like a cosmological constant; for shorter time scales, we no longer get acceleration). On general grounds, such a scalar field must have a mass of order $m_\phi^2 \sim H_0^2$. Second, the Compton wavelength of this scalar field is on the order of the Hubble distance, so it will mediate an attractive fifth force which is distinguishable from gravity by the absence of any coupling to light. Unless the coupling to matter is tiny compared to that of gravity, many solar system based tests of gravity would fail, such as measurements of the bending of light around the Sun [15,16].

The archetypal example of $f(R)$-DE suffers from problems such as these. This model invokes the function [39]

$$f(R) = R - \frac{\mu^4}{R} \tag{1}$$

for $\mu \approx H_0$. This gives a $V(\phi)$ in the EF with a runaway exponential potential at late times: $V(\phi) \sim H_0^2 M_{\rm pl}^2 \exp(-(3/\sqrt{6})\phi/M_{\rm pl})$ (here large ϕ means small R which means late times.) With no matter, such a potential in the JF gives rise to an accelerating universe with the equation of state parameter $w_X = -2/3$ [39]. This model, however, is riddled with problems. First, the theory does not pass solar system tests [20–22,29], and second, the cosmology is inconsistent with observation when nonrelativistic (NR) matter is present [23]. Both problems can be understood in the dual scalar-tensor theory.

For cosmology, during the matter-dominated phase but at high redshifts, the influence on the dynamics of ϕ from the potential V is small compared to the influence from the coupling to matter, which manifests itself in terms of an effective potential for ϕ of the form

$$V_{\rm eff}(\phi) = V(\phi) + \bar{\rho}_{\rm NR} \exp\left(-\frac{\phi}{\sqrt{6}M_{\rm pl}}\right), \tag{2}$$

where $\bar{\rho}_{\rm NR}$ is the energy density of NR matter. (More details of the exact form of this potential will be presented in the next section.) The second term dominates because $H_0^2 M_{\rm pl}^2 \ll \bar{\rho}_{\rm NR}$, and ϕ then rolls down the potential generated by $\bar{\rho}_{\rm NR}$ and not V. The result is that the universe is driven away from the expected matter-dominated era (MDE) into a radiation dominated expansion in the JF with $H^2 \propto a^{-4}$, after which it crosses directly into the accelerating phase, with expansion driven by DE with an effective equation of state parameter $w = -2/3$. This special radiation-dominated-like phase (which is not driven by radiation) was dubbed the ϕMDE by [23], where it was made clear that this phase is inconsistent with observation. We say that this potential V is unstable to large cosmological nonrelativistic densities.

For the solar system tests, the potential $V(\phi)$ is also negligible, so the theory behaves exactly like a scalar-tensor theory with no potential. Because the coupling to matter has the same strength as that to gravity, the scalar field mediates a long-range fifth force, and the theory is ruled out by solar system tests. In particular, [22] found that $\gamma = 1/2$ in the parametrized post-Newtonian (PPN) framework, which is in gross violation of the experimental bound.

The above solar system tests also seem to rule out more general classes of $f(R)$-DE models [19,29–31,34]. However on the cosmology front, it seems that one *can* cook up examples of $f(R)$ consistent with some dynamical dark energy [24,25,27]: by demanding that the cosmological expansion $a(t)$ takes a certain form, one can integrate a differential equation for the function f that by design gives a universe with any desired expansion history $a(t)$. In this way, one gets around the cosmological instability of the archetypal model mentioned above. However, these functions are arguably very contrived, and further investigation of solar system predictions is required to determine whether these models are viable.

D. What $f(R)$ theories are allowed?

We now try to find viable $f(R)$ theories by examining what *is* acceptable on the scalar-tensor side. We focus on theories that pass solar system tests. Because the coupling of the scalar field to matter is fixed in $f(R)$ theories, and the only freedom we have is with the potential V, we must choose V in such a way as to hide the scalar field from the solar system tests that caused problems for the models described above. We are aware of only two ways to do this. The first is the Chameleon scalar field, which uses nonlinear effects from a very specific singular form of potential to hide the scalar field from current tests of gravity [40,41]. The second is simply to give the scalar field a quadratic potential with mass $m_\phi \gtrsim 10^{-3}$ eV, so that the fifth force has an extent less than 0.2 mm and so cannot be currently measured by laboratory searches for a fifth force [42].

We will find simple $f(R)$ models which reproduce these two types of potentials and so by design pass solar system tests. Finding functions f which give *exactly* these potentials will simply generate models which are indistinguishable from their scalar-tensor equivalent. However, if we search for simple choices of f that reproduce these potentials in a certain limit, then these theories will not be exactly equivalent and might have distinguishable features.

The Chameleon type $f(R)$ model seems to be the most plausible model for attacking DE, as at first glance it seems to get around the general problems mentioned above. Indeed, one Chameleon model will arise quite naturally from a simple choice of f. However, we will show that the solar system constraints on this model preclude any possible interesting late-time cosmological behavior: the acceleration is observationally indistinguishable from a cosmological constant. In particular, for all the relevant physical situations this Chameleon model is the same as has been considered before with no distinguishing features. However, this model might provide clues in a search for viable $f(R)$ theories that pass solar system tests and that may give interesting late-time behavior.

In an independent recent analysis, [30,43] also discussed the Chameleon effect in $f(R)$ theories. They focus on a slightly different set of Chameleon potentials and come to similar conclusions. Their results and ours together suggest that the Chameleon effect may be generic to $f(R)$ theories.

We now turn from attempts to explain DE in $f(R)$ models to an arguably more plausible scenario, which is simply to give the scalar field a large mass. These models have no relevance for dynamic DE, but they do have interesting consequences for early-universe cosmology. The most theoretically best motivated functions, namely, polynomials in R, fit this class of $f(R)$ theories. The aim of this investigation is to explore what we can possibly know about the function f. Because this question is very general, we will restrict our attention to a subclass of plausible $f(R)$ models.

For these polynomial models, we will investigate possible inflationary scenarios where the scalar partner ϕ is the inflaton. We find the relevant model parameters which seed the fluctuations of the cosmic microwave background (CMB) in accordance with the experiment. We then investigate general constraints on the model parameters where ϕ is not an inflaton. We use solar system tests, nucleosynthesis constraints, and finally an instability which is present in these theories when another slow-roll inflaton ψ is invoked to explain CMB fluctuations. This instability is analogous to that of the ϕMDE described above.

The rest of this paper is organized as follows. In Sec. II we review the equivalence of $f(R)$ theories with scalar-tensor theories, elucidating all the essential points we will need to proceed. Then in Secs. III and IV we explore the Chameleon model and massive theories, respectively, focusing on observational constraints. We summarize our conclusions in Sec. V.

II. $f(R)$ DUALITY WITH SCALAR-TENSOR THEORIES

We study the "modified" gravity theory defined by the action

$$S_{JF} = \int d^4x\sqrt{-g}\frac{M_{\rm pl}^2}{2}f(R) + S_M[g_{\mu\nu}, \Psi, A_\alpha, \ldots], \quad (3)$$

where, for example, $\Psi, A_\alpha, \ldots$ label the matter fields of the standard model. Here we present a rundown of the map to the scalar-tensor theory, displaying the most important points needed to proceed. See, for example, [11,38,44] for more details of the equivalence with scalar-tensor theories.

We *choose* to fix the connection in R as the Christoffel symbols and not an independent field, as opposed to the Palatini formalism, which results in a very different theory [45–49].

If one simply varies the action equation (3) with respect to the metric $g_{\mu\nu}$, then a fourth order equation for the metric results. One can argue (using general coordinate invariance) that the degrees of freedom in the field $g_{\mu\nu}$ can be split into a massless spin-2 field $\tilde{g}_{\mu\nu}$ and a massive scalar field ϕ with second order equations of motion. This split is easily revealed at the level of the action. Following, for example, [44] we introduce a new auxiliary scalar field Q (a Lagrange multiplier). The gravity part of Eq. (3) may be written as

$$S_{\rm grav} = \int d^4x\sqrt{-g}\frac{M_{\rm pl}^2}{2}(f'(Q)(R-Q)+f(Q)). \tag{4}$$

As long as $f''(Q) \neq 0$, the equation of motion $(\delta/\delta Q)$ gives $Q = R$ and Eq. (4) becomes the original gravity action. This may be written in the more suggestive form

$$S_{\rm grav} = \int d^4x\sqrt{-g}\left(\frac{M_{\rm pl}^2}{2}\chi R - \chi^2 V(\chi)\right) \tag{5}$$

by relabeling $f'(Q) \equiv \chi$. This is a scalar-tensor theory of gravity with Brans Dicke parameter $\omega_{\rm BD} = 0$ [50] and potential [44]

$$V(\chi) = \frac{M_{\rm pl}^2}{2\chi^2}[Q(\chi)\chi - f(Q(\chi))]. \tag{6}$$

Here $Q(\chi)$ solves $\chi = f'(Q)$. Finally a rescaling of the metric (which should be thought of as a field relabeling)

$$\tilde{g}_{\mu\nu} = \chi g_{\mu\nu} = e^{(2/\sqrt{6})\phi/M_{\rm pl}}g_{\mu\nu} \tag{7}$$

reveals the kinetic terms for the scalar field

$$S_{\rm EF} = \int d^4x\sqrt{-\tilde{g}}\left(\frac{M_{\rm pl}^2}{2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) + S_M[\tilde{g}_{\mu\nu}e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_{\rm pl}}}, \Psi, A_\alpha, \ldots], \tag{8}$$

where the new canonical scalar field ϕ is related to χ, Q, R through

$$f'(R) = f'(Q) = \chi = \exp(\sqrt{2/3}\phi/M_{\rm pl}). \tag{9}$$

As the kinetic terms for $\tilde{g}_{\mu\nu}$ and ϕ are now both canonical, we see that these are the true degrees of freedom of $f(R)$ gravity. This demonstrates that the theories defined by $S_{\rm JF}$ (the Jordan frame) and $S_{\rm EF}$ (the Einstein frame) are completely equivalent when $f''(Q) \neq 0$. We choose to analyze the theory in the Einstein frame as things are much simpler here. It is, however, important to be careful to interpret results correctly, making reference to what is observed. In particular, matter is defined in the Jordan frame, and hence it will be most sensible to talk about JF observables. We will give a simple example of this when we have introduced some matter.

The equations of motion for ϕ resulting from Eq. (8) are

$$-\tilde{\Box}\phi = -\frac{dV}{d\phi} - \frac{\tilde{T}^M}{\sqrt{6}M_{\rm pl}}, \tag{10}$$

and for the metric $\tilde{g}_{\mu\nu}$,

$$\tilde{R}_{\mu\nu} - \tfrac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = M_{\rm pl}^{-2}(\tilde{T}^M_{\mu\nu} + \tilde{T}^\phi_{\mu\nu}) \tag{11}$$

with the energy-momentum tensors

$$\tilde{T}^M_{\mu\nu} = \chi^{-1}T^M_{\mu\nu}(\chi^{-1}\tilde{g}_{\mu\nu}\ldots), \tag{12}$$

$$\tilde{T}^\phi_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + \tilde{g}_{\mu\nu}(-\tfrac{1}{2}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)). \tag{13}$$

Note that only the combination $\tilde{T}^M_{\mu\nu} + \tilde{T}^\phi_{\mu\nu}$ is conserved in the EF.

There are two important observations to be made about Eq. (8) relating to the extra coupling to matter. First, the $\tilde{T}^M/M_{\rm pl}\sqrt{6}$ term in Eq. (10) represents an additional density-dependent "force" on the scalar field, and for special cases where we can solve for the functional form of the ϕ dependence of $\tilde{T}^M/M_{\rm pl}\sqrt{6}$ explicitly, as in [41], we can think of the scalar field living in an effective potential. We will see two examples where this force is important, the most dramatic being the Chameleon effect.

Second, ϕ couples to matter as strongly as conventional gravity ($\tilde{g}_{\mu\nu}$) does. Hence, as was already mentioned, ϕ will mediate a detectable fifth force for solar system tests unless we do something dramatic to hide it. Finding theories which hide ϕ from solar system tests is the focus of this paper.

A. Matter and cosmology in $f(R)$ theories

Let us first consider the coupling to standard model fields, assuming that they are defined in the JF. This is important for understanding how ϕ may decay. Massless scalar fields conformally coupled to gravity and massless gauge bosons behave the same in the two frames and so do not couple to ϕ. However, a minimally coupled (real) scalar field Φ and a Dirac field Ψ have extra interactions with ϕ in the EF:

$$S_\Phi = \int d^4x\sqrt{-\tilde{g}}\left\{-\frac{1}{2}(\partial\tilde{\Phi})^2 - \frac{1}{2}m_\Phi^2\chi^{-1}\tilde{\Phi}^2 - \frac{1}{12M_{\rm pl}^2}\tilde{\Phi}^2\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{\sqrt{6}M_{\rm pl}}\tilde{\Phi}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\phi\right\}, \tag{14}$$

$$S_\Psi = \int d^4x\sqrt{-\tilde{g}}\bar{\tilde{\Psi}}(i\tilde{\gamma}^\mu\tilde{D}_\mu - m_\Psi\chi^{-1/2})\tilde{\Psi}, \tag{15}$$

where the JF fields have been rescaled as $\tilde{\Phi} = \chi^{-1/2}\Phi$ and $\tilde{\Psi} = \chi^{-3/4}\Psi$. Note that the cosmologically evolving field $\phi = \bar{\phi}(t)$ will change the masses of the standard model particles in the EF as

$$\tilde{m} = m\chi^{-1/2} = m\exp(-(\sqrt{1/6})\bar{\phi}(t)/M_{\rm pl}) \tag{16}$$

and small excitations $\delta\phi$ around the average value $\bar{\phi}(t)$ will roughly speaking interact via the vertices defined by the interaction Lagrangian,

$$\frac{1}{\sqrt{6}M_{\rm pl}}(\tilde{m}_\Phi^2\delta\phi\tilde{\Phi}^2 - \tilde{\Phi}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\delta\phi + \tilde{m}_\Psi\delta\phi\bar{\tilde{\Psi}}\tilde{\Psi}) \tag{17}$$

to lowest order in $1/M_{\rm pl}$. The mass shift in Eq. (16) has an interesting consequence in the EF; it shifts the frequency of the absorption and emission lines by a factor of $\chi^{-1/2}$. This effect will be indistinguishable from the normal cosmological redshift due to expansion, and our effective redshift will be the combination of both cosmological expansion and mass shift: $(1+z)^{-1} = \tilde{a}\chi^{-1/2}$, where $\tilde{a}$ is the scale factor in the EF normalized equal to unity today. This combination turns out to be the Jordan frame scale factor a (see below), so our redshift measurements coincide in both frames as expected. These ideas were recently discussed in the context of conformal cosmology [51,52], where the observed redshifts are explained completely in terms of an evolving scalar field.

Perfect fluids are best examined in the JF, because it is here that their energy-momentum tensor is conserved. For a general JF metric one can solve for the flow of the fluid using conservation of $T^M_{\mu\nu}$ and number flux nU^μ (or other relevant physical principles) and then map into the EF via Eq. (12).

1. The homogeneous and isotropic case

For example, consider a homogeneous isotropic cosmology,

$$\text{(JF)}\qquad ds^2 = dt^2 - a(t)^2 d\vec{x}^2, \qquad U^\mu = (\partial_t)^\mu, \tag{18}$$

$$\text{(EF)}\qquad d\tilde{s}^2 = d\tilde{t}^2 - \tilde{a}(\tilde{t})^2 d\vec{x}^2, \qquad \tilde{U}^\mu = (\partial_{\tilde{t}})^\mu, \tag{19}$$

where U^μ and $\tilde{U}^\mu$ are the local fluid velocities in the two frames. The quantities above are related by

$$\tilde{a} = \chi^{1/2}a, \qquad d\tilde{t} = \chi^{1/2}dt, \qquad \tilde{U}^\mu = \chi^{-1/2}U^\mu. \tag{20}$$

These relations imply that the Hubble parameters in the two frames are related by

$$H = \chi^{1/2}\left(\tilde{H} - \frac{\tilde{\partial}_t\phi}{\sqrt{6}M_{\rm pl}}\right). \tag{21}$$

For example, applying the principles of entropy "conservation" and number conservation in the JF (one may also need to demand thermal and chemical equilibrium as relevant to the early universe) results in known functions $\rho(a)$ and $p(a)$ such that the EF energy-momentum tensor may be written as

$$\tilde{T}^M_{\mu\nu} = \tilde{\rho}\tilde{U}_\mu\tilde{U}_\nu + \tilde{p}(\tilde{U}_\mu\tilde{U}_\nu + \tilde{g}_{\mu\nu}), \tag{22}$$

where

$$\tilde{\rho} = \chi^{-2}\rho(\tilde{a}\chi^{-1/2}), \qquad \tilde{p} = \chi^{-2}p(\tilde{a}\chi^{-1/2}). \tag{23}$$

The cosmological equations of motion are

$$3\tilde{H}^2M_{\rm pl}^2 = \tilde{\rho} + \tfrac{1}{2}(\tilde{\partial}_t\phi)^2 + V(\phi), \tag{24}$$

$$\tilde{\partial}_t^2\phi + 3\tilde{H}\tilde{\partial}_t\phi = -\frac{\partial V_{\rm eff}(\phi,\tilde{a})}{\partial\phi} = -\frac{dV_E}{d\phi} - \frac{\tilde{T}^M}{\sqrt{6}M_{\rm pl}}. \tag{25}$$

The effective potential for the scalar field coupled to homogeneous and isotropic matter is

$$V_{\rm eff}(\phi,\tilde{a}) = V(\phi) + \tilde{\rho} = V(\phi) + \chi^{-2}\rho(\tilde{a}\chi^{-1/2}). \tag{26}$$

For the special case where the only density components present are nonrelativistic ($\rho = \rho_{\rm NR} \propto a^{-3}$) and ultrarelativistic ($\rho = \rho_{\rm R} \propto a^{-4}$) fluids, the effective potential is

$$V_{\rm eff}(\phi) = V(\phi) + \bar{\rho}_{\rm NR}(\tilde{a})e^{-(\phi/M_{\rm pl}\sqrt{6})} + \bar{\rho}_{\rm R}(\tilde{a}), \tag{27}$$

where for convenience we define $\bar{\rho}_{\rm NR}(\tilde{a}) \equiv \chi^{-3/2}\rho_{\rm NR}(\tilde{a}\chi^{-1/2}) \propto \tilde{a}^{-3}$ and $\bar{\rho}_{\rm R}(\tilde{a}) \equiv \chi^{-2}\rho_{\rm R}(\tilde{a}\chi^{-1/2}) \propto \tilde{a}^{-4}$. These expressions are now independent of ϕ: all the ϕ dependence is explicitly seen in Eq. (27). Note that relativistic particles provide no force on ϕ because $\tilde{T}$ vanishes, or equivalently because $\bar{\rho}_R(\tilde{a})$ appears simply as an additive constant to the potential in Eq. (27).

2. The spherically symmetric case

We now turn to the case of a spherically symmetric distribution of nonrelativistic matter $\rho_{\rm NR}(r)$ in the JF, for which we aim to solve for the metric $g_{\mu\nu}$. We wish to consider this problem in the EF, where ϕ will take a spherically symmetric form and gravity behaves like GR coupled to $\tilde{\rho} = \chi^{-2}\rho_{\rm NR}$. In the weak field limit, we write the metrics in the two frames as

$$\text{(JF)}\qquad ds^2 = -(1-2A(r))dt^2 + (1+2B(r))dr^2 + r^2d\Omega^2, \tag{28a}$$

$$\text{(EF)}\qquad d\tilde{s}^2 = -(1-2\tilde{A}(\tilde{r}))dt^2 + (1+2\tilde{B}(\tilde{r}))d\tilde{r}^2 + \tilde{r}^2d\Omega^2, \tag{28b}$$

where $\tilde{r} = \chi^{1/2}r$ and for small $\phi/M_{\rm pl}$, the gravitational potentials are related by

$$A(r) \approx \tilde{A}(\tilde{r}) + \frac{\phi(\tilde{r})}{\sqrt{6}M_{\rm pl}}, \tag{29a}$$

$$B(r) \approx \tilde{B}(\tilde{r}) + \frac{1}{\sqrt{6}M_{\rm pl}}\frac{d\phi(\tilde{r})}{d\ln\tilde{r}}. \tag{29b}$$

Following [41] we define a nonrelativistic energy density $\bar{\rho}_{\rm NR}(\tilde{r}) = \chi^{-3/2}\rho(r)$ in the EF which is conserved there and is analogous to $\bar{\rho}_{\rm NR}(\tilde{a})$ defined above for cosmology. Ignoring the backreaction of the metric on ϕ, we take $\tilde{g}_{\mu\nu} \approx \eta_{\mu\nu}$ in Eq. (10) and find as in [41] that

$$\frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d\phi}{d\tilde{r}}\right) = V'(\phi) - \chi^{-(1/2)}\frac{\bar{\rho}_{\rm NR}(\tilde{r})}{\sqrt{6}M_{\rm pl}} = \frac{\partial V_{\rm eff}(\phi,\tilde{r})}{\partial\phi}, \tag{30}$$

where again the effective potential is

$$V_{\rm eff} = V(\phi) + \chi^{-1/2}\bar{\rho}_{\rm NR}(\tilde{r}). \tag{31}$$

Solving Eq. (30) for ϕ then allows us to find the metric in the JF via Eq. (29).

As an instructive example, consider the quadratic potential $V(\phi) = m_\phi^2\phi^2/2$ and a uniform sphere of mass M_c and radius R_c. The solution external to the sphere is given by a Yukawa potential

$$\frac{\phi(r)}{M_{\rm pl}} = \frac{1}{\sqrt{6}}\frac{M_c e^{-m_\phi r}}{4\pi M_{\rm pl}^2 r} \tag{32}$$

assuming that $m_\phi R_c \ll 1$ and $\phi/M_{\rm pl} \ll 1$ so that $\tilde{r} \approx r$. If we ignore the energy density of the profile $\phi(r)$, then outside the object there is vacuum. The metric in the EF is then simply the Schwarzschild solution for mass M_c. In other words, the potentials in Eq. (28a) are given by $\tilde{A}(\tilde{r}) = \tilde{B}(\tilde{r}) = M_c/8\pi M_{\rm pl}^2\tilde{r}$ in the weak field limit $|\tilde{A}|, |\tilde{B}| \ll 1$. In the JF using Eq. (29), one finds the corresponding potentials

$$A(r) \approx \tilde{A}(r)[1 + \tfrac{1}{3}e^{-m_\phi r}], \tag{33a}$$

$$B(r) \approx \tilde{A}(r)[1 - \tfrac{1}{3}e^{-m_\phi r}(1 + m_\phi r)]. \tag{33b}$$

For $r \ll m_\phi^{-1}$ we find that the PPN parameter $\gamma = 1/2$, a well-known result for a Brans Dicke theory [50] with $\omega_{BD} = 0$ [15].

The key feature here is the effective potential from Eqs. (27) and (31). We have now seen that it makes a crucial difference in two situations, and it will play an important role in the next two sections as well.

III. AN $f(R)$ CHAMELEON

In this section, we consider $f(R)$ theories that are able to pass solar system tests of gravity because of the so-called "Chameleon" effect. We first present a theory that is by design very similar to the original Chameleon model presented in [40]. We will give a brief description of how this model evades solar system constraints, and then move on to the cosmology of these $f(R)$ theories, concentrating, in particular, on their relation to DE. Throughout this discussion we refer the reader to the original work [40,41,53,54], highlighting the differences between the original and $f(R)$ Chameleons.

The Chameleon model belongs to the following general class of models,

$$f(R) = R - (1-m)\mu^2\left(\frac{R}{\mu^2}\right)^m - 2\Lambda. \tag{34}$$

The sign of the second factor is important to reproduce the Chameleon, and the $(1-m)$ factor ensures that the theory is equivalent to GR as $m \to 1$. These models have been considered before in the literature [11,23]; in particular, this class contains the original DE $f(R)$ of Eq. (1) when $m = -1$, $\Lambda = 0$, and $2\mu^4 \to \mu^4$.

The potential for ϕ in the EF is

$$V(\phi) = \frac{M_{\rm pl}^2\mu^2}{2\chi^2}(m-1)^2\left(\frac{\chi - 1}{m^2 - m}\right)^{m/(m-1)} + \frac{M_{\rm pl}^2\Lambda}{\chi^2}, \tag{35}$$

where $\chi = \exp(\sqrt{2/3}\phi/M_{\rm pl})$ as usual. For $0 < m < 1$ and for $|\phi/M_{\rm pl}| \ll 1$, this reduces to

$$V(\phi) = M^{4+n}(-\phi)^{-n} + M_{\rm pl}^2\Lambda, \tag{36}$$

defined for $\phi < 0$, where the old parameters μ, m are related to the new parameters M, n through

$$m = \frac{n}{1+n}, \qquad \mu^2 = \frac{(2(1+n)^2)^{1+n}}{(\sqrt{6}n)^n}\frac{M^{4+n}}{M_{\rm pl}^{n+2}}. \tag{37}$$

The preferred values used in [40] are $M \sim 10^{-3}$ eV and $n \sim 1$. In the $f(R)$ theory, these values give $m \sim 1/2$ and $\mu \sim 10^{-50}$ eV, i.e. much smaller than the Hubble scale today.

For small $|\phi|/M_{\rm pl}$, this singular potential is equivalent to the potential considered in [40] for the Chameleon scalar field, albeit with $\phi \to -\phi$. The coupling to matter, which is a very important feature of this model, is also very similar. In [40], a species of particles i is assumed to have its own Jordan frame metric $g_{\mu\nu}^{(i)}$, with respect to which it is defined, and a conformal coupling to the metric in the EF

$$g_{\mu\nu}^{(i)} = e^{2\beta_i\phi/M_{\rm pl}}\tilde{g}_{\mu\nu}. \tag{38}$$

Comparing this to Eq. (7), the $f(R)$ Chameleon has $\beta_i = -1/\sqrt{6}$ for all matter species, so that all the Jordan frame metrics coincide.

In the original Chameleon model, the β_i were specifically chosen to be different so that ϕ would show up in tests of the weak equivalence principle (WEP). The $f(R)$ Chameleon does not show up in tests of the WEP, so the solar system constraints will be less stringent here.

This coupling to matter, along with the singular potential equation (36), are the defining features of this $f(R)$ that make it a Chameleon theory. The effective potential $V_{\rm eff}$, discussed in the previous section (see, for example, Eq. (27)), is then a balance between two forces; V pushing ϕ toward more negative values and the density-dependent term pushing ϕ toward more positive values. So although the singular potential equation (36) has no minimum and hence no stable "vacuum," the effective potential equation (27) including the coupling to matter does have a minimum. In fact, the density-dependent term pushes the scalar field ϕ up against the potential wall created by the

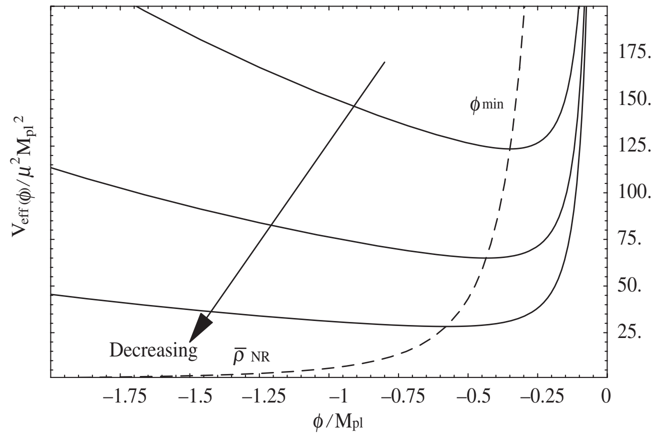

FIG. 1. Effective potential for the Chameleon model equation (35) with decreasing $\bar{\rho}_{\rm NR}/\mu^2 M_{\rm pl}^2 = 100$, 50, 20 and $m = 1$. Note that $\phi_{\rm min}$ and the mass m_ϕ^2 (the curvature of the minimum) are very sensitive to the background energy density $\bar{\rho}_{\rm NR}$.

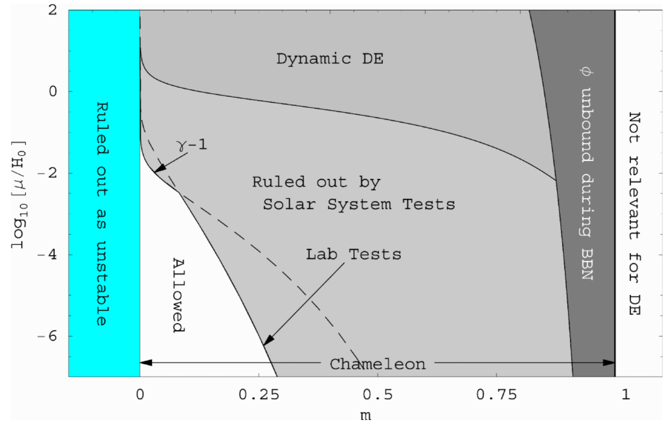

FIG. 2 (color online). Solar system constraints on the $f(R)$ Chameleon are seen to exclude all models where the "dark energy" is observationally distinguishable from a cosmological constant (labeled "dynamic DE"). The two different solar system constraint curves come from Eqs. (46) and (47). Although it is not clear from the plot, the limits $m \to 0$, $m \to 1$, and $\mu \to 0$ are all acceptable and yet give no dynamical DE. Indeed these are exactly the limits in which we recover standard GR.

singularity in V at $\phi = 0$. Indeed, the field value $\phi_{\rm min}$ at the minimum of the effective potential $V_{\rm eff}$ and the mass m_ϕ of ϕ's excitation around that given minimum are both highly sensitive increasing functions of the background density $\bar{\rho}_{\rm NR}$, as illustrated in Fig. 1. Using Eq. (31) for small $|\phi|/M_{\rm pl}$, the field value at the minimum and the curvature of the minimum are, respectively,

$$-\frac{\phi_{\rm min}}{\sqrt{6}M_{\rm pl}} = \frac{m(1-m)}{2}\left(\frac{M_{\rm pl}^2\mu^2}{\bar{\rho}_{\rm NR}}\right)^{1-m}, \tag{39}$$

$$m_\phi^2 = \frac{2}{3(1-m)}\frac{\bar{\rho}_{\rm NR}}{M_{\rm pl}^2}\left(-\frac{\sqrt{6}M_{\rm pl}}{\phi_{\rm min}}\right). \tag{40}$$

It is plausible that a scalar field ϕ which is very light for cosmological densities is heavy for solar system densities and hence currently undetectable. However, as we will now see, the actual mechanism that "hides" ϕ from solar system tests is a bit more complicated than this.

A. Solar system tests

In this section, we will derive solar system and laboratory constraints on the parameters (μ, m), summarized in Fig. 2. The profile of $\phi(\tilde{r})$ in the solar system (around the Earth, around the Sun, etc.) is of interest for solar system tests of gravity: it determines the size of the fifth force and the post-Newtonian parameter γ. Because the effective potential for ϕ changes in different density environments, the differential equation governing the profile $\phi(\tilde{r})$ in Eq. (30) is highly nonlinear, and the standard Yukawa profile of Eq. (32) does not always arise. These nonlinear features have been studied in [41], where it was found that for a spherically symmetric object of mass M_c and radius R_c surrounded by a gas of asymptotic density ρ_∞, the profile is governed by the so-called "thin-shell" parameter,

$$\Delta = \frac{|\phi_{\rm min}^\infty - \phi_{\rm min}^c|}{\sqrt{6}M_{\rm pl}}\frac{24\pi M_{\rm pl}^2 R_c}{M_c}, \tag{41}$$

where $\phi_{\rm min}^\infty$ and $\phi_{\rm min}^c$ are the minima of the effective potential in the presence of the asymptotic energy density $\bar{\rho}_{\rm NR} = \rho_\infty$ and central energy densities $\bar{\rho}_{\rm NR} = \rho_c$, respectively, see Eq. (39). If Δ is large, then the external profile of ϕ is the usual Yukawa profile equation (32) with mass $m_\infty^2 = m_\phi^2$, the curvature of the effective potential in the presence of the asymptotic density $\bar{\rho}_{\rm NR} = \rho_\infty$; see Eq. (40). If Δ is small, then the Yukawa profile is suppressed by a factor of Δ. The term "thin shell" comes from the fact that only a portion of such a thin-shell object contributes to the external Yukawa profile, the thickness of the shell being roughly (ΔR_c). We simply treat Δ as a parameter that suppresses this profile if $\Delta \ll 1$.

For example, let us consider the profile ϕ around the Sun, with $M_c = M_{\rm Sun}$ and $R_c = R_{\rm Sun}$. Assuming that we are in the thin-shell regime ($\Delta \ll 1$), the Yukawa profile of Eq. (32) suppressed by a factor Δ becomes

$$\phi(r) = \frac{\Delta}{\sqrt{6}}\frac{M_{\rm Sun}e^{-m_\infty r}}{4\pi M_{\rm pl} r} + \phi_{\rm min}^\infty. \tag{42}$$

As in [40], we take the asymptotic density used to find ϕ_∞ and m_∞ as that of the local homogeneous density of dark and baryonic matter in our Galaxy: $\rho_\infty \approx 10^{-24}$ g/cm^3. Following the discussion in Sec. II, the metric in the EF external to the Sun is just the Schwarzschild metric (in the weak field limit) with Newtonian potential $\tilde{A}(r) \approx M_{\rm Sun}/(8\pi M_{\rm pl}^2 r)$. Using Eq. (29) to map this metric into the JF metric $g_{\mu\nu} = \chi^{-1}\tilde{g}_{\mu\nu}$, we find

$$ds^2 = -\left[1 - 2\tilde{A}(r)\left(1 + \frac{\Delta}{3}e^{-m_\infty r}\right)\right]dt^2 + r^2 d\Omega^2 + \left[1 + 2\tilde{A}(r)\left(1 - \frac{\Delta}{3}e^{-m_\infty r}(1 + m_\infty r)\right)\right]dr^2. \quad (43)$$

Assuming that the Compton wavelength m_∞^{-1} is much larger than solar system scales (we will confirm this later), we obtain within the PPN formalism [15] that

$$\gamma = \frac{3 - \Delta}{3 + \Delta} \approx 1 - (2/3)\Delta. \quad (44)$$

There are several observational constraints on $|\gamma - 1|$, including ones from the deflection of light and from Shapiro time delay. The tightest solar system constraint comes from Cassini tracking, giving $|\gamma - 1| \lesssim 2.3 \times 10^{-5}$ [16]. Thus the thin-shell parameter satisfies $\Delta \lesssim 3.5 \times 10^{-5}$. We note that $|\phi_{\rm min}^c| \ll |\phi_{\rm min}^\infty|$ because of the sensitive dependence of $\phi_{\rm min}$ on the local density, so the definition of Δ in Eq. (41) becomes

$$\Delta \approx 3|\phi_{\rm min}^\infty|/\sqrt{6}M_{\rm pl}\tilde{A}(R_{\rm Sun}), \quad (45)$$

where $\tilde{A}(R_c = R_{\rm Sun}) \approx 10^{-6}$ is the Newtonian potential at the surface of the Sun. Using Eq. (39) with $\bar{\rho}_{\rm NR} = \rho_\infty \approx 10^{-24}$ g/cm^3 gives the constraint

$$\frac{\mu^2}{H_0^2} \lesssim 3\left(\frac{2}{m(1 - m)}\right)^{1/(1-m)} 10^{((-6-5m)/(1-m))} \quad (46)$$

on the theory parameters μ and m. For theories which fail this bound, we find that the Compton wavelength of ϕ for the asymptotic background density of our Galaxy satisfies $m_\infty^{-1} \gtrsim 10^{10}$ AU. This confirms the assumption that m_∞^{-1} is large compared to solar system scales, which was used to derive this bound.

As was already noted, the solar system constraints derived in [41] are more restrictive. This is because they demanded that the couplings (β_i) to different species of particles in Eq. (38) be different. This gives violations of the weak equivalence principle on Earth-based experiments unless the Earth and atmosphere have a thin shell. However, in the $f(R)$ Chameleon model, all the β_i are the same, so there will be no weak equivalence principle violations.

The $f(R)$ Chameleon may still show up in searches for a fifth force, in particular, in tests of the inverse square law. The strongest comes from Earth-based laboratory tests of gravity such as in the Eöt-Wash experiments [42]. By demanding that the test masses acquire thin shells, [41] found constraints on the parameters (M, n) which map into the following bound on the $f(R)$-parameters (μ, m):

$$\frac{\mu^2}{H_0^2} \lesssim (1 - m)\left(\frac{2}{m(1 - m)}\right)^{(m/(1-m))} 10^{((-4-24m)/(1-m))}. \quad (47)$$

B. Cosmology

We now turn to the cosmology of the Chameleon scalar field, which was studied in [53]. It was found there and already commented on in [41] that the mass of ϕ on cosmological scales is not small enough to give any interesting DE behavior for $M \approx 10^{-3}$ eV and $n \sim 1$. We will revisit this question in the $f(R)$ context: do any *allowed* parameters (μ, m) in Eqs. (46) and (47) give nonvanilla DE? Will there be any cosmologically observable differences between this $f(R)$ Chameleon and the original model (which is in principle possible because higher order terms in the expansion of V in Eq. (35) may become important)? We will see that the answer to both of these questions is *no* for the same reason: solar system tests preclude the minimum of the effective potential from lying beyond $\phi \lesssim -M_{\rm pl}$ on cosmological scales today.

Let us try to understand this by looking at the details of Chameleon cosmology. We first note that, as opposed to [53], we do *not* fix $\Lambda M_{\rm pl}^2 = M^4$, so we are less restricted as to what M or μ can be. The essence of the argument, however, is the same as in [53]. Working in the EF, for a large set of initial conditions in the early universe, ϕ is attracted to the minimum of the effective potential given by Eq. (27). The scalar field tracks the minimum, which shifts $\phi(\tilde{a}) \equiv \phi_{\rm min}$ as the universe expands. The energy density in coherent oscillations around this minimum are negligible and so there is no "moduli problem." (In contrast, this *may* be a problem for the case considered below in Sec. IV.)

We will see that the condition for such a tracking solution to be valid is that the minimum satisfies

$$-\phi(\tilde{a})/M_{\rm pl} \ll 1, \quad (48)$$

so we consistently make this assumption to derive properties of the tracking minimum. After matter-radiation equality we have the tracking solution

$$-\frac{\phi(\tilde{a})}{\sqrt{6}M_{\rm pl}} = \frac{m(1 - m)}{2}\left(\frac{M_{\rm pl}^2\mu^2}{\bar{\rho}_{\rm NR}(\tilde{a}) + 4V(\phi(\tilde{a}))}\right)^{1-m}. \quad (49)$$

Along this tracking solution, the curvature (mass) around the minimum and the speed of the minimum are, respectively,

$$\frac{m_\phi^2(\tilde{a})}{\tilde{H}^2} = \frac{2}{1 - m}\left(\frac{\sqrt{6}M_{\rm pl}}{-\phi(\tilde{a})}\right)\left(\frac{\bar{\rho}_{\rm NR}(\tilde{a}) + 4V(\phi(\tilde{a}))}{\bar{\rho}_{\rm NR}(\tilde{a}) + V(\phi(\tilde{a}))}\right), \quad (50)$$

$$\frac{-1}{M_{\rm pl}\tilde{H}}\frac{d\phi(\tilde{a})}{d\tilde{t}} = -3\left(\frac{-\phi(\tilde{a})}{M_{\rm pl}}\right)\frac{(1 - m)\bar{\rho}_{\rm NR}(\tilde{a})}{\bar{\rho}_{\rm NR}(\tilde{a}) + 4V(\phi(\tilde{a}))}. \quad (51)$$

Since ϕ will track the minimum while $m_\phi(\tilde{a}) \gg \tilde{H}$, Eq. (50) shows that the assumption of Eq. (48) is indeed consistent.

Also, during radiation domination one can show that $m_\phi^2(\tilde{a})/\tilde{H}^2 \sim (-M_{\rm pl}/\phi(\tilde{a}))\tilde{a}/\tilde{a}_{\rm MR}$, where $\tilde{a}_{\rm MR}$ is the scale factor at matter-radiation equality, so it is possible that at early times the scalar field is unbound. We know the expansion history and the effective value of Newton's constant G_N quite well [39,55] around big bang nucleosynthesis (BBN); if ϕ is unbound, we have no reason to believe that G_N, which varies as ϕ varies, is near today's value. Requiring that it is bound before the beginning of BBN gives a constraint that we have included in Fig. 2.

Returning to the matter-dominated era, Eq. (48) implies that the expansion history in the JF may be written as

$$3M_{\rm pl}^2H^2 = \rho_{\rm NR}(a) + V(\phi(a_0)) + \mathcal{O}\left(\frac{\phi}{M_{\rm pl}}\right). \tag{52}$$

For $|\phi(a_0)|/M_{\rm pl} \ll 1$ today, this is just the usual Friedmann equation with a cosmological constant, where in accordance with the experiment we are forced to identify $V(\phi(a_0))$ with $\rho_X(0)$, the current dark energy density. Note that the parameter Λ in V, which we have not fixed, allows us to make this choice independent of any values of μ and m. For m not small, $V(\phi(a_0)) \approx \Lambda M_{\rm pl}^2$ so Λ is fixed at $\Lambda \approx \rho_X(0)/M_{\rm pl}^2$; however, for small m we will see later that the situation will be slightly different.

This implies that the only way to get interesting late-time cosmological behavior is to not have $|\phi(a_0)/M_{\rm pl}| \ll 1$ but rather $|\phi(a_0)/M_{\rm pl}| \sim 1$ today. In this case the tracking solution above is not valid; the scalar field is no longer stuck at the minimum, and we might not have to invoke a constant Λ in V to explain today's accelerated expansion. Rather, the acceleration would be driven by a quintessence type phase.

However, one can show that given the solar system constraints, $|\phi(a_0)/M_{\rm pl}| \sim 1$ is not possible. In fact, as we will now show, a stronger statement can be made: even if we continue to assume Eq. (48), so that the tracker solution is still valid, the solutions that are consistent with solar system tests always give DE behavior that is "vanilla," i.e., indistinguishable from a cosmological constant.

In these models, the effective dark energy density is

$$\rho_X(a) \approx V(\phi(a)) + \left(\frac{-\phi(a)}{\sqrt{6}M_{\rm pl}}\right) \times (\rho_{\rm NR}(a) + \rho_X(0)) \times \left(2 + \frac{6\rho_{\rm NR}(a)(1-m)}{\rho_{\rm NR}(a) + 4\rho_X(0)}\right), \tag{53}$$

where $V(\phi(a)) - \rho_X(0) = \mathcal{O}(\phi/M_{\rm pl})$. If we expected Eq. (53) to give interesting behavior in the allowed region of parameter space, we would fit the Friedmann equation with $\rho_X(a)$ to the combined knowledge of the expansion history and find the allowed values of (μ, m). We will instead adopt a simpler approach, *defining* "nonvanilla DE" through the effective equation of state parameter,

$$w_X = -\frac{1}{3}\frac{d\ln\rho_X(a)}{d\ln a} - 1. \tag{54}$$

This is the relevant equation of state that one would measure from the expansion history (that is *not* p_ϕ/ρ_ϕ). We say that the DE is nonvanilla if $|w_X + 1| > 0.01$, which is quite optimistic as to future observational capabilities [56]. However, because our result is null the exact criterion is not important.

The resulting constraint on μ and m is shown in Fig. 2 along with the solar system constraints. As the figure shows, all models consistent with solar system tests are "vanilla"—that is, indistinguishable from a cosmological constant.

The most interesting part of this parameter space is the limit $m \to 0$, which is one of the limits in which we should recover general relativity. The theory then becomes

$$f(R) \approx R - (\mu^2 + 2\Lambda) + \mu^2 m \ln(R/\mu^2) \tag{55}$$

with the Chameleon-like (singular at $\phi = 0$) potential

$$V(\phi) \approx \frac{M_{\rm pl}^2}{2}e^{-(4/\sqrt{6})(\phi/M_{\rm pl})}(\mu^2 + 2\Lambda - m\mu^2\ln(1 - e^{(\sqrt{2/3})(\phi/M_{\rm pl})})). \tag{56}$$

In this limit we are forced to fix $\Lambda = \rho_X(0)/M_{\rm pl}^2 - \mu^2/2$. The DE energy equation of state parameter is $w_X = -1 - 0.05\ m\mu^2/H_0^2$. The tightest solar system constraint on μ^2 in this limit is from $|\gamma - 1|$ in Eq. (46) which gives $m\mu^2 \lesssim 6 \times 10^{-6}H_0^2$. The equation of state parameter for DE is then constrained to be $|w_X + 1| \lesssim 0.3 \times 10^{-6}$ which is definitely unobservable.

Finally we note that the ultimate fate of the $f(R)$ chameleon is different from that of the original model. This is because $V(\phi)$ actually does have a minimum relevant for cosmological energy densities. This is due to the ϕ dependence of the $\Lambda M_{\rm pl}^2\chi^{-2}$ term in Eq. (35), which is absent in the original models. Eventually ϕ will settle into this minimum and the universe will enter an inflating de Sitter phase, much like the fate of a universe with a simple cosmological constant. The original model on the other hand *eventually* enters a quintessence-like expansion. However, this distinction is unobservable today.

In conclusion to this section, we have found a previously unstudied class of $f(R)$ theories that gives acceptable local gravity by exploiting the Chameleon effect. For the allowed parameters of this model, there is no interesting late-time cosmological behavior (observably dynamic DE). That is not to say that these models have no interesting physics—there may indeed be some interesting effects of such models for future solar system tests [41] or on large scale structure [57], and this might warrant further study in the context of $f(R)$ models. We also noted that the $f(R)$ model is subtly different from the original Chameleon model. It does not violate the weak equivalence principle, so solar system constraints are less stringent and the ultimate fate of the universe is now simply an inflating de Sitter spacetime.

This mechanism might also be a starting point for constructing working modified gravity models which do give nonvanilla DE, somehow exploiting this mechanism more effectively and bridging the gap in Fig. 2 between solar system constraints and nonvanilla DE. We suspect they will not be as simple as the one presented. This mechanism may also be relevant for attempting to understand the Newtonian limit of the artificially constructed $f(R)$ models mentioned earlier that reproduce an exact expansion history. We make this claim because an important property of the model presented in [27] is that the parameter $B \propto f''(R)$ is a rapidly growing function of the scale factor a. For small $f''(R)$, one can show that the mass curvature of V is $m_\phi^2 \sim 1/f''(R)$. Hence, in this theory the mass of the scalar field during cosmological evolution is large at early times and small at late times, as in the Chameleon models. A more detailed analysis, beyond the scope of this paper, is required to see whether nonlinear effects play a part in the Newtonian limit of these theories.

IV. MASSIVE $f(R)$ THEORIES

We now consider arguably more realistic $f(R)$ theories, namely, polynomials $f(R) = -2\Lambda + R + aR^2 + bR^3 \ldots$ These theories have been extensively studied, especially for quadratic $f(R)$; see [36] and references therein. They are more natural from the point of view of renormalization and effective field theories: a high energy completion of gravity would allow us to find these higher order terms. However, common wisdom would have the higher order terms suppressed by inverse powers of $M_{\rm pl}$ and would force us to include other terms of the same mass dimension such as $R^{\mu\nu}R_{\mu\nu}$. Despite this, we wish to explore the phenomenology of such polynomial $f(R)$ theories and hence constrain them with cosmological observations. In doing so, we will explore the full range of values for the coefficients $(a, b, \ldots)$ of the higher order terms to be conservative rather than assume that they are of order unity in Planck units.

This class of theories can only match the currently observed cosmic acceleration via an explicit cosmological constant term $f(0) = -2\Lambda$, giving the identification $\Lambda = \rho_X(0)/M_{\rm pl}^2 = 3H_0^2\Omega_\Lambda$, so there is no hope of dynamical DE. Rather, these theories are more relevant to very early-universe cosmology where R is large, and hence some of our results will be quite speculative.

Consider for simplicity the two-parameter model

$$f(R) = R + R\left(\frac{R}{\mu^2}\right) + \lambda R\left(\frac{R}{\mu^2}\right)^2. \tag{57}$$

We restrict to the parameter range $\mu^2 > 0$ and $0 < \lambda < 1/3$, so that the resulting potential V has a stable quadratic minimum and is defined for all ϕ. The Einstein frame potential for ϕ or χ is given by

$$V_E(\chi) = \frac{M_{\rm pl}^2\mu^2}{2\chi^2}q^2(1 + 2\lambda q), \tag{58}$$

where

$$q \equiv \frac{1}{3\lambda}\left[\sqrt{1 - 3\lambda(1-\chi)} - 1\right] \tag{59}$$

is the larger of the two roots of $1 - \chi + 2q + 3\lambda q^2$ (this ensures that the resulting potential has a stable minimum). We plot this potential for various λ in Fig. 3.

We will first explore the possibility that ϕ is the inflaton, then discuss other constraints from our knowledge about the early universe. Figure 4 summarizes our constraints.

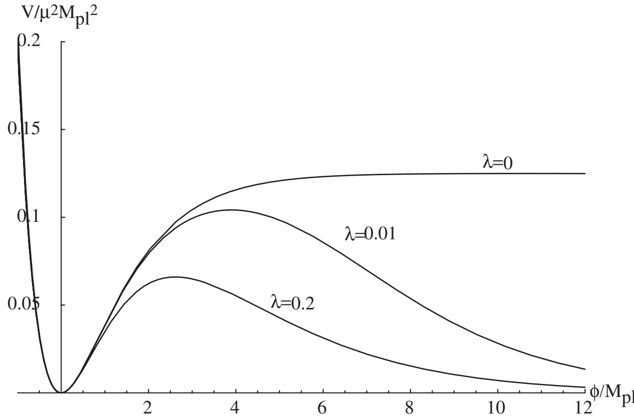

FIG. 3. Potential for the $f(R)$ model in Eq. (57) with various values of λ. Notice that the $\lambda = 0$ case has an asymptotically flat potential as $\phi \to \infty$.

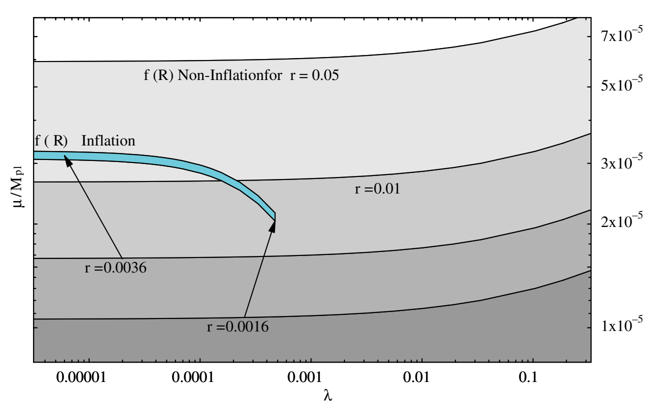

FIG. 4 (color online). Constraints on the cubic $f(R)$ model. The thin gray (blue) sliver corresponds to observationally allowed $f(R)$ inflationary scenarios. Shaded are regions we may rule out given a measurement of the tensor to scalar ratio r and the assumption that they were generated by a period of slow-roll inflation in the early universe. The $r = 0.05$ and $r = 0.01$ are the most realistic curve, in the sense that future experiments are sensitive to such values as low as $r = 0.01$ [56].

A. $f(R)$ inflation

The possibility that higher order corrections to the gravitational Lagrangian might be responsible for a de Sitter inflationary period was examined thoroughly early on in the inflationary game [12,58]. For $\lambda = 0$, the potential $V(\phi)$ is very flat for large ϕ, which is perfect for inflation. This model and other related ideas were extensively studied in [59–64], which also confirmed the existence of a viable inflationary model. We now search for possible inflationary scenarios with $\lambda \neq 0$ that are consistent with current observations. This question was already considered in [65], which found $\lambda \ll 1$; however, we wish to be more quantitative in light of the latest CMB measurements.

As usual in these models, it is important to keep careful track of whether we are working in the EF or the JF: recall that the potential V is defined in the EF, while matter is most naturally considered in the JF. Nonetheless, we will argue that the inflationary predictions are exactly the same as those of general relativity plus a normal slow rolling scalar field with potential $V(\phi)$. The argument goes as follows. Slow-roll inflation works normally in the EF where the graviton and scalar field have canonical actions. In particular, the EF is where one *should* calculate the spectrum of tensor and scalar mode fluctuations. Reheating and the transformation of fluctuations in ϕ to adiabatic density fluctuations also works as usual in the EF, because at this time the cosmic fluid is relativistic and hence governed by the same equations of motion in both frames. After reheating, ϕ is frozen out at the minimum of V, with $\phi = 0$ and $\chi = 1$, so there is no longer any distinction between the JF and the EF ($\tilde{g}_{\mu\nu} = g_{\mu\nu}$). Calculations for $\lambda = 0$ were performed both as above and in the JF in [61], and the results were found to be consistent as expected.

Using this idea, calculating the inflationary predictions is straightforward. Using Eq. (17), we can estimate the reheating temperature as $T_{\rm RH} \approx 1.3 \times 10^{-2} g_*^{-1/4}(N_s\mu^3/M_{\rm pl})^{1/2}$, where N_s is the number of minimally coupled scalar fields into which ϕ decays (it decays most strongly into these fields). Then the scale factor (normalized to $a = 1$ today) is

$$a_{\rm end} = 7.5 \times 10^{-32}\left(\frac{\mu}{M_{\rm pl}}\right)^{-1/6} g_*^{-1/12} N_s^{1/6} \tag{60}$$

at the end of inflation. Integrating the slow-roll equations of motion, $\phi' = -V'(\phi)/3\tilde{H}$, and assuming $\lambda \ll 1$, the number of e-foldings of inflation for a mode k is

$$N_k \approx \frac{3\,\mathrm{arctanh}(\sqrt{\lambda}q)}{2\sqrt{\lambda}} - \frac{3}{4}\ln(1 + 2q_k) + N_0(\lambda). \tag{61}$$

Here N_0 is a small number defined such that $N_k(q_{\rm end}) = 0$ at the end of inflation, where $q = q_{\rm end} \approx 1/\sqrt{3}$, and q_k is related to the conformal factor $\chi_k = 1 + 2q_k + 3\lambda q_k^2$ when the mode k crosses the horizon:

$$\tilde{H}_k \approx \mu/\sqrt{24} = k e^{N_k}/a_{\rm end}. \tag{62}$$

This particular mode will have a scalar fluctuation amplitude (also referred to as δ_H^2 in the literature)

$$Q_k^2 = \frac{1}{1200\pi^2\epsilon_k}\left(\frac{\mu^2}{M_{\rm pl}^2}\right), \tag{63}$$

where the slow-roll parameters (using the definitions in [66]) are

$$\epsilon_k \approx \frac{(1 - \lambda q_k^2)^2}{3q_k^2}, \qquad \eta_k \approx -\frac{2(1 + \lambda q_k^2)}{3q_k}. \tag{64}$$

We then use these to find the scalar spectral index $n_s = 1 - 6\epsilon + 2\eta$, the ratio of tensor to scalar modes $r = 16\epsilon$, etc. Using the combined WMAP + SDSS measurements [2] $Q = (1.945 \pm 0.05) \times 10^{-5}$ for modes $k = 0.002/\mathrm{Mpc}$ we can use Eqs. (61)–(63) together to fix μ. For $\lambda \to 0$ the result is

$$\mu \approx (3.2 \pm 0.1) \times 10^{-5}\, M_{\rm pl}, \tag{65}$$

$$n_s \approx 0.964, \tag{66}$$

$$r \approx 0.0036, \tag{67}$$

which is consistent with both the theoretical results of [60,61] and recent observational constraints [1,2].

In addition, n_s is sensitive to the value of λ. The observational constraint $0.937 < n_s < 0.969$ (68% C.L.) from [2], translates into a strong upper bound on λ:

$$\lambda < 4.7 \times 10^{-4}. \tag{68}$$

This is an example of the usual fine-tuning that is needed for observationally allowed inflationary potentials and is consistent with the findings of [65]. More precisely the values of μ, λ appropriate for inflation are shown in Fig. 4.

B. Other constraints

Above we explored the possibility that ϕ was the inflaton. Let us now turn to the alternative possibility that ϕ is not the inflaton, and compute miscellaneous constraints on the parameters μ and λ when they are varied freely. We will first consider the fifth force mediated by ϕ, then investigate how the scalar field behaves dynamically in the early universe, where the most interesting effect comes from considering a period of slow-roll inflation driven by some *other* scalar field. As noted in Sec. II, the dynamics of ϕ is still governed by an effective potential equation (27) which is important when there is a component of matter whose energy-momentum tensor has nonzero trace.

To begin with, we ignore any effect that such a term may have on the minimum of $V_{\rm eff}$ for these polynomial models, which is a good approximation if $|\tilde{T}^\mu_\mu| \ll \mu^2 M_{\rm pl}^2$. We will see that for the first few constraints that we derive, this will indeed be the case. Then we will return to the question of where this is a bad approximation, which will naturally lead to our discussion of slow-roll inflation by some other scalar field.

1. Fifth force constraints

The minimum of the effective potential lies at $\chi = 1$, $\phi = 0$. The curvature of this minimum is $m_\phi^2 = \mu^2/6$. Hence we can get around solar system constraints simply by making μ large enough so that the range of the fifth force will be small. Clearly it must have a range smaller than the solar system, otherwise, as was discussed above, it will violate the bound on the PPN parameter γ. (Recall that there is no Chameleon effect here, so $\Delta = 1$ in Eq. (43) and $\gamma = 1/2$.) For smaller scales, we consider searches for a fifth force via deviations from the inverse square law. The profile for a quadratic potential, i.e., Eq. (32), gives a Yukawa potential between two test masses m_1 and m_2:

$$V(r) = -\alpha \frac{m_1 m_2}{8\pi M_{\rm pl}^2} \frac{e^{-m_\phi r}}{r}, \tag{69}$$

where $\alpha = 1/3$. For this α value, a fifth force is ruled out for any Compton wavelength m_ϕ^{-1} ranging from solar system scales down to 0.2 mm, where the lower bound comes from the Eöt-Wash experiments [42]. This bound translates to

$$\mu \gtrsim 1.0 \times 10^{-3}\ \text{eV}. \tag{70}$$

This implies $V(\phi) \sim \mu^2 M_{\rm pl}^2 \gg \rho_{\rm solar}$, a typical solar system density, so for this constraint we were justified in ignoring any effects of the density-dependent term on the minimum of $V_{\rm eff}$.

2. Nucleosynthesis constraints

Given this preliminary constraint from local gravity tests, let us now consider the cosmology of ϕ in the EF. We may approximate the potential around the minimum by a quadratic potential $V_{\rm eff}(\phi) \approx (\mu^2/12)\phi^2$, which is valid for $|\phi| \lesssim M_{\rm pl}$. The interesting behavior will come during the radiation dominated epoch, so in Eq. (24) we take $\tilde\rho(\tilde a) \approx \bar\rho_R(\tilde a) \propto \tilde a^{-4}$, and we ignore the $\tilde T^\mu_\mu$ term in Eq. (25) to find the cosmological equations of motion

$$3\tilde H^2 M_{\rm pl}^2 = \bar\rho_R(\tilde a) + \frac{\mu^2}{12}\phi^2 + \frac{1}{2}(\phi')^2, \tag{71}$$

$$\phi'' + 3\tilde H \phi' + \frac{\mu^2}{6}\phi = 0, \tag{72}$$

where the primes denote $d/d\tilde t$. There are two interesting limiting behaviors, corresponding to $\tilde H \gg \mu$ and $\tilde H \ll \mu$, which we will now explore in turn.

For $\tilde H \gg \mu$, the friction term in Eq. (72) dominates, and ϕ is frozen out at some value ϕ_* with $d\phi/d\tilde t = 0$. The energy density of ϕ is subdominant in Eq. (71). Therefore, in the EF we have the usual radiation dominated expansion, and in the JF using Eqs. (21) and (23) we have the same Friedmann-Robertson-Walker (FRW) expansion with a different effective Newton's constant G_N^*: $3H^2 = 8\pi G_N^* \rho(a) \propto a^{-4}$, where

$$G_N^* = \frac{1}{8\pi M_{\rm pl}^2}\exp\left(-\sqrt{\frac{2}{3}}\frac{\phi_*}{M_{\rm pl}}\right). \tag{73}$$

For $\tilde H \ll \mu$, on the other hand, assuming $\phi_* < M_{\rm pl}$, the field ϕ starts to oscillate with frequency $\mu/\sqrt{6}$ and an amplitude that redshifts as $\tilde a^{-3/2}$. Hence in the EF, the energy density of ϕ in Eq. (71) from these zero momentum field oscillations is $\rho_\phi = (\mu^2/12)\phi^2 + \phi'^2/2 \approx \rho_\phi^*(\tilde a/\tilde a_*)^{-3}$, where $\rho_\phi^* \approx (\mu^2/12)\phi_*^2$. Mapping back into the JF, and averaging over a cycle of this oscillation, we obtain the Friedmann equation

$$3H^2 M_{\rm pl}^2 = \rho_R(a) + \tfrac{3}{2}\rho_\phi^*(a/a_*)^{-3}, \tag{74}$$

where the unusual factor of 3/2 comes from the averaging of the oscillations of G_N^* in Eq. (73), as is discussed in more depth in [67].

The crossover between these two behaviors occurs when $\tilde H$ is comparable to μ, and given the laboratory tests of gravity above we can say that this must occur when the universe has at least the temperature $T_* \gtrsim 1$ TeV. We were therefore justified in assuming radiation domination in our calculation.

Let us examine further the zero momentum oscillations of ϕ that give this extra nonrelativistic energy density. In the absence of some mechanism (such as an extra period of low scale inflation [68]), we expect the initial amplitude of oscillations to be of the order $M_{\rm pl}$. This is because the potential in Fig. 5 varies on the scale of $M_{\rm pl}$ independently of the height of V. Hence in the absence of any other scale, the initial amplitude must be around this size. Recall that at the onset of oscillations, $\tilde H \sim \mu$, so the initial energy

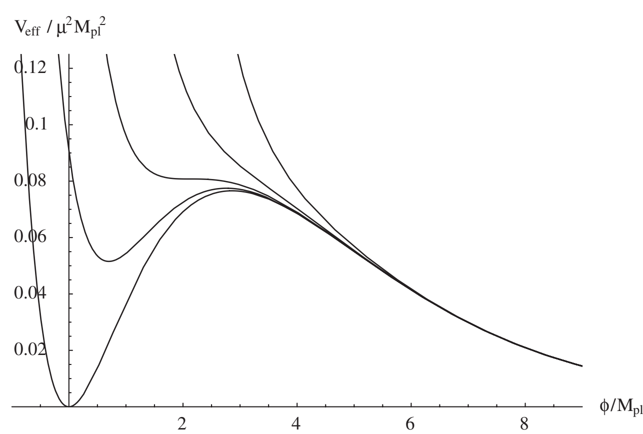

FIG. 5. Effective potential for the polynomial model equation (57) with $\lambda = 0.1$ for various JF inflationary energy densities $U(\psi)$.

density of these oscillations is

$$\rho_\phi^* \sim M_{\rm pl}^2 \mu^2 \sim \tilde{H}^2 M_{\rm pl}^2 \sim \rho_R(a_*). \tag{75}$$

This energy density subsequently grows relative to the radiation density component, quickly forcing the universe into a matter-dominated period of expansion. This is unacceptable if this component does not decay before the onset of BBN, because then at the time of BBN the expansion would be much faster than the normal radiation dominated expansion, which would be inconsistent with observed primordial abundances [55].

The fact that ϕ interacts weakly with other particles (the vertices in Eq. (17) are suppressed by $1/M_{\rm pl}$) so that ϕ decays too slowly is exactly what is known as the cosmological *moduli problem*. To be more precise, we can use Eq. (17) to estimate the decay rate of zero momentum modes into other massive particles:

$$\Gamma_\phi \approx \sum_s \left(\frac{m_s^4}{m_\phi M_{\rm pl}^2 96\pi} - \frac{m_\phi m_s^2}{96\pi M_{\rm pl}^2} + \frac{m_\phi^3}{M_{\rm pl}^2 384\pi} \right) + \sum_f \frac{m_f^2 m_\phi}{M_{\rm pl}^2 12\pi}, \tag{76}$$

where the sums are over minimally coupled scalar particles and fermions with masses $2m_s$, $2m_f < m_\phi$. The requirement $\Gamma_\phi > H_{\rm BBN}$ translates into the constraint $\mu \gtrsim 100$ TeV for the standard model. One would expect the bound on μ to be slightly smaller if one includes other particles that have not been detected yet with mass smaller than 100 TeV. This constraint should not be taken too seriously, however, because the moduli problem may hypothetically be resolved by electroweak scale inflation [69] or even by a brief second period of inflation at the electroweak scale [68].

3. Density-dependent forces

We now consider how the extra density-dependent term in $V_{\rm eff}$ may affect cosmology. In other words, when can we not neglect the forcing term $\tilde{T}^\mu_\mu$ of Eq. (25)? After ϕ enters the oscillating phase when $\mu \gg \tilde{H}$, the extra term has little effect on the minimum since then it is small compared to the size of the potential itself ($V \sim \mu^2 M_{\rm pl}^2$). As a result, ϕ simply oscillates as expected. Before the crossover, when ϕ is frozen, we showed that the universe must be radiation dominated so that, in particular, as $\tilde{T}^\mu_\mu \ll \tilde{\rho} = 3\tilde{H}^2 M_{\rm pl}^2$ during this phase, the Hubble friction will dominate compared to the force term of $\tilde{T}^\mu_\mu$ in Eq. (25), and we were justified in claiming that ϕ is frozen out. The cosmology here does not suffer from the instability that plagues Eq. (1).

There are, however, some exceptions that might lead to interesting constraints. First, consider a relativistic component i of the cosmological plasma that becomes nonrelativistic and dumps its energy into the other relativistic components. In this case, $-\tilde{T}^\mu_\mu \sim (g_i/g_*)\tilde{\rho}$ for a period of about one e-folding, so ϕ receives a kick and is displaced by an amount $\Delta\phi \approx (g_i/g_*) M_{\rm pl}/\sqrt{6}$ [53]. This might lead to an interesting effect such as ϕ being kicked out of the basin of attraction of V. The extreme case would be that ϕ does not end up oscillating around the minimum as expected when $\tilde{H} \sim \mu$, but instead ends up rolling down the tail of V, an effect which is clearly only possible for $\lambda \neq 0$. In principle, such kicks could even invalidate the predictions of BBN: near the onset of BBN, $e^\pm$ annihilation occurs, displacing ϕ and consequently changing G_N^* significantly as per Eq. (73). However, we have already shown that ϕ must be in the oscillatory phase long before the onset of BBN, and we have argued that these kicks have no effect while ϕ is in the oscillator phase, so in fact this effect is unlikely to have relevance for BBN. Such kicks may affect other important cosmological dynamics at temperatures higher than $T > 1$ TeV, such as baryogenesis. However, the effects are extremely model dependent, and it is hard to say anything definitive at this point.

4. Noninflation

Another situation when we cannot ignore the density-dependent force on ϕ is during inflation. Here $\tilde{T}^\mu_\mu$ is large for many e-foldings. Remember that in this section, we are not considering ϕ as our inflaton; instead we consider a slow-roll inflationary period driven by some other scalar field ψ defined in the JF. We wish to examine the effect a modified gravity Lagrangian such as Eq. (57) has on the inflationary scenario. In particular, we will be interested in situations where inflation by the field ψ does not work, being effectively sabotaged by ϕ. We will discuss the generality of these assumptions at the end.

Such models have been considered before in the context of both the $\lambda = 0$ models [70–72] and other generalized gravity models [73]. There the goal was generally to make the inflationary predictions more successful, focusing on working models.

In the JF, consider a scalar field ψ with a potential $U(\psi)$. We assume that ψ is slow rolling; $d\psi/dt \approx -U'(\psi)/3H(t)$. This is the assumption that

$$\frac{d^2\psi}{dt^2} \ll U'(\psi), \qquad \left(\frac{d\psi}{dt}\right)^2 \ll U(\psi), \tag{77}$$

which must be checked for self-consistency once we have solved for $H(t)$. We can now easily calculate $H(t)$ by first working in the EF and mapping back to the JF. The equations of motion in the EF, Eqs. (24) and (25), become

$$3\tilde{H}^2 M_{\rm pl}^2 = \tfrac{1}{2}\phi'^2 + V_{\rm eff}(\phi), \qquad \phi'' + 3\tilde{H}\phi' = -V'_{\rm eff}(\phi), \tag{78}$$

$$V_{\rm eff}(\phi) = V(\phi) + U(\psi)\chi^{-2}. \tag{79}$$

It is interesting that a constant vacuum term in the JF does not translate into a constant term in the EF. See Fig. 5 for some examples of the effective potential $V_{\rm eff}$; we see that for large enough $U(\psi) \gg \mu^2 M_{\rm pl}^2$, the minimum vanishes. One finds that there is no minimum of the effective potential for

$$U(\psi) > \frac{\mu^2 M_{\rm pl}^2}{18\sqrt{3\lambda}}. \tag{80}$$

In particular, there is always a minimum for $\lambda = 0$.

The resulting behavior of the inflaton ψ depends on the size of $U(\psi)$ compared to $\mu^2 M_{\rm pl}^2$. For small $U(\psi) \ll \mu^2 M_{\rm pl}^2$, it is clear that there is a stable minimum around which ϕ will oscillate. In this situation, the effective potential has a minimum at $\phi \approx 0$ with value approximately $V_{\rm eff}(\phi \approx 0) \approx U(\psi)$, so after the energy density of ϕ oscillations redshift away, we are left with an exponentially expanding universe with $\chi \approx 1$, $3\tilde{H}^2 M_{\rm pl}^2 \approx U(\psi)$, and $\tilde{H} \approx H$. Hence in the JF, gravity behaves as it normally would in general relativity: for a flat potential, the slow-roll conditions are satisfied, and inflation driven by ψ works as it normally would. This is the expected situation, and it will happen for $\mu \approx M_{\rm pl}$.

On the other hand, we now show that when $U(\psi) \gg \mu^2 M_{\rm pl}^2$ and when there is no minimum of the effective potential ($\lambda \neq 0$), we get a contradiction to the assumption that ψ was slow rolling. Hence we show that it is not possible for ψ to drive slow-roll inflation. For large $U(\psi) \gg \mu^2 M_{\rm pl}^2$, the potential may be approximated as $V_{\rm eff} \approx U(\psi)\chi^{-2}$. We treat $U(\psi)$ as a constant and find that there is an exact attractor solution to Eq. (78) of the form $\chi \sim \tilde{t}$ and $\tilde{a} \sim \tilde{t}^{3/4}$. Mapping this into the EF, we find the behavior $a \sim t^{1/2}$, i.e., a period of radiation dominated expansion analogous to the ϕMDE of [23]. More specifically, we find

$$3M_{\rm pl}^2 H^2 \approx U(\psi) a^{-4}. \tag{81}$$

This is clearly not an inflating universe. So the slow-roll assumptions of Eq. (77) are not consistent in this case. We therefore conclude that it is not possible for ψ to drive slow-roll inflation.

Instead, ψ dumps most of its energy $U(\psi_0)$ into radiation, and as before, ϕ is left frozen at some point ϕ_* until $U(\psi_0)\tilde{a}^{-4} \sim \mu^2 M_{\rm pl}^2$. After this, ϕ can either drive an inflationary period itself as in the original discussion of $f(R)$ inflation, or if ϕ_* is not in the basin of attraction of $V_{\rm eff}$, it will roll down the tail of $V_{\rm eff}$. In neither situation has ψ inflated our universe. From this combination of inflaton ψ plus $f(R)$ gravity with $U(\psi) \gg \mu^2 M_{\rm pl}^2$, we only get satisfactory inflation if (μ, λ) lies in the region of parameter space appropriate for $f(R)$ inflation (the gray (blue) sliver in Fig. 4) and if ϕ_* sits at a point which allows for the required number of e-foldings.

5. Gravitational wave constraints

It is well known that inflation produces horizon-scale gravitational waves of amplitude $Q_t \sim H/M_{\rm pl}$, so that the energy scale of inflation can be bounded from above by the current observational upper limit $Q_t \lesssim 0.6 \times 10^{-5}$ [1,2] and perhaps measured by a detection of the gravitational wave signal with future CMB experiments [56]. Using such a detection one might try to constrain μ^2 by the arguments of the previous section. Specifically, by demanding that during inflation there is a minimum of the effective potential one can find a constraint by invoking Eq. (80) with $U(\psi)$ (incorrectly) replaced by the measured energy density of inflation.

However, because of the EF-JF duality, one needs to carefully define what one means by "the energy scale of inflation." The bound from the above argument simply precludes inflatons with a given energy density $U(\psi)$ in the JF, but $U(\psi)$ is merely a parameter which does not necessarily set the energy scale of inflation. In addition to this problem, we cannot use Eq. (80) to derive a constraint for $\lambda = 0$, because in this case there is always a minimum in the effective potential and it is always possible for ψ to slow roll (this situation is described in greater depth in [70,71]).

To make these ideas more concrete and resolve both of these ambiguities, we will operationally define the energy scale of inflation to be the one that makes the standard GR formula for the gravitational wave amplitude valid. It is clear that the amplitude of gravitational waves should be calculated in the EF where the metric has a canonical action. The result is then passed trivially into the JF after inflation and when $\phi = 0$. The Hubble scale $\tilde{H}$ then sets the size of the fluctuations, but it is a complicated model dependent calculation to find exactly when the relevant fluctuations are generated. However, there is a limit to the size of $\tilde{H}$ for which the EF is approximately inflating, and so gravitational waves are being generated. Following the discussion above of nonworking inflatons, we demand that ϕ must be slow rolling down the effective potential $V_{\rm eff}$ defined in Eq. (79) for both frames to be inflating. In this situation, both scalar and gravity modes are being generated.

The procedure is thus to find the maximum value of $\tilde{H}$ (that is, from Eq. (78), the maximum value of $V_{\rm eff}$) such that ϕ is slow rolling. We then maximize this $\tilde{H}$ with respect to the parameter $U(\psi)$ to find the largest amplitude of gravitational waves that can possibly be produced. At each step in this procedure, we wish to be as conservative as possible; for example, we define slow roll through the slow-roll parameter constraints $\epsilon < 1$ and $|\eta| < 2$ to allow for the possibility of power law inflation. Where again we use the standard defintion of η and ϵ from [66].

As an example, consider the $\lambda = 0$ case. Here it is possible to show that for ϕ to be slow rolling, it must satisfy

$$\phi > \phi_{\rm sr} \equiv \sqrt{\frac{3}{2}}M_{\rm pl}\ln\left(\frac{2}{3} + \sqrt{\frac{7}{9} + \frac{8U(\psi)}{3M_{\rm pl}^2\mu^2}}\right), \tag{82}$$

where $\phi_{\rm sr}$ always lies to the left of the minimum of $V_{\rm eff}$. The maximum Hubble scale in the EF for a given $U(\psi)$ is then $\tilde{H}^2 < \max\{V_{\rm eff}(\phi_{\rm sr})/3M_{\rm pl}^2, \mu^2/24\}$. This is maximized for large $U(\psi)/\mu^2M_{\rm pl}^2$, with the result that $\tilde{H}^2 < \mu^2/6$ where we have used Eq. (82). This translates into a constraint on the maximum gravitational wave amplitude that can be produced,

$$Q_t^{\rm max} \approx 0.04\frac{\mu}{M_{\rm pl}} \Rightarrow r^{\rm max} \approx 5\times 10^6\frac{\mu^2}{M_{\rm pl}^2}. \tag{83}$$

Given a measurement of the tensor to scalar ratio r, this places a limit on μ:

$$\mu \gtrsim 3\times 10^{-4}r^{1/2}M_{\rm pl}. \tag{84}$$

Numerically, we find similar results for nonzero λ. We plot examples of this constraint in Fig. 4, combined with the already discussed working $f(R)$ inflationary models. Note that for a given r, it is important that this constraint lies below the corresponding working $f(R)$ inflationary model (the gray (blue) thin sliver of Fig. 4) with the same r; fortunately, as is indicated by the arrows in this figure, it does.

If gravitational waves are not detected, then this argument gives no lower bound on μ. In particular, it is possible that inflation occurred at the electroweak scale, in which case the constraint $\mu \gtrsim 2\times 10^{-3}$ eV is the best we can do.

Note that we completely ignore the production of scalar fluctuation modes for this argument. This is because the scalar modes are much more difficult to calculate, since there are two scalar fields in the mix, ψ and ϕ, which are canonically defined in different frames. But the scalar modes are also model dependent and one should generally be able to fine-tune $U(\psi)$ to give the correct amplitude and spectral index without affecting the above argument. This more complicated problem was considered for chaotic inflation with R^2 gravity in [72].

This constraint applies only to slow-roll inflation models. There are classes of fast-roll inflation, but these models have problems of their own and generally fail to reproduce the required scale invariance (see [74] for a review).

Finally, let us discuss some inflaton models that might circumvent this constraint. It is possible to add an inflaton in the EF. However, this theory is then not conformally equivalent to an $f(R)$ theory: the two scalar fields $\tilde{\psi}$ and ϕ get mixed up. Hence it is not in the class of models we set out to constrain. Another possibility is to add an inflaton which is conformally coupled to gravity and has a $V \propto \psi^4$ potential. This does not change from frame to frame and so inflation might be expected to work. However, it was shown by [75] that nonminimally coupled scalar fields cannot drive inflation.

In any case, if gravitational waves are found, then this constraint must be thought about seriously when using such $f(R)$ models in other astrophysical or local gravity situations.

V. CONCLUSIONS

We have searched for viable $f(R)$ theories using the wealth of knowledge on scalar-tensor theories to which $f(R)$ theories are equivalent. We studied two classes of models: the $f(R)$ Chameleon and massive $f(R)$ theories, which may well be the only classes of models that can be made consistent with local gravity observations.

The $f(R)$ Chameleon that was studied is a special kind of scalar field which hides itself from solar system tests of gravity using nonlinear effects associated with the all-important density-dependent effective potential. It was shown that, despite this Chameleon behavior, solar system tests still preclude the possibility of observably dynamical DE; the best we could do was $|w_X - 1| \lesssim 0.3\times 10^{-6}$ for the effective DE equation of state parameter w_X relevant for the dynamics of the expansion. There are of course interesting effects of the Chameleon both for local gravity [40] and on cosmological density perturbations [57], and these may be worth future studies in the context of $f(R)$ theories.

The massive theories were found to be more relevant for very high energy cosmology, so the conclusions were more speculative. First, the scalar field may be the inflaton, in which case we found the required polynomial $f(R)$ to be quite fine-tuned as is usual for inflationary potentials. If the scalar field was not the inflaton, then we saw that possible instabilities could spoil both inflation and big bang nucleosynthesis, giving interesting constraints on the shape of $f(R)$. If primordial gravitational waves are detected using the CMB, then the most naive models of inflation have serious problems unless the mass of the $f(R)$ scalar is very large; a measured scalar to tensor ratio of $r = 0.05$ requires $\mu \gtrsim 7\times 10^{-5}$ $M_{\rm pl}$. If gravitational waves are not found, then the best we can say comes from the Eöt-Wash laboratory experiments constraining the extent of a 5th force: $\mu \gtrsim 2\times 10^{-3}$ eV.

General relativity adorned with nothing but a cosmological constant, i.e., $f(R) = R - 2\Lambda$, is a remarkable successful theory. As we have discussed, a host of observational data probing scales from 10^{-2} m to 10^{26} m not only agree beautifully with GR, but also place sharp constraints on the parametrized departures from GR that we have explored. In particular, both viable classes of $f(R)$ theories that we studied were found to have no relevance for dynamic dark energy that is observationally distinguishable from vanilla dark energy, i.e., a cosmological constant. Since we have no good reason to believe that there are additional viable classes of $f(R)$ theories, it appears likely that no viable $f(R)$ theories can produce the sort of interesting nonvanilla dark energy that many observers are hoping to find. However, without a much larger study of the parameter space (which is of course incredibly large) we shy away from making a stronger statement here.

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