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INVERSE LIMITS WITH SET VALUED FUNCTIONS

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ABSTRACT. We begin to answer the question of which continua can be homeomorphic to an inverse limit with a single upper semi-continuous bonding map from $[0,1]$ to $2^{[0,1]}$. Several continua including $[0,1] \times [0,1]$ and all compact manifolds with dimension greater than one cannot be homeomorphic to such an inverse limit. It is also shown that if the upper semi-continuous bonding maps have only zero dimensional point values, then the dimension of the inverse limit does not exceed the dimension of the factor spaces.

1. INTRODUCTION

Recent work by Mahavier and Ingram [2] has raised interest in inverse limits with upper semi-continuous set valued functions. With all inverse limits it is important to study how the structures of the inverse limit are determined by the factor spaces and the bonding maps. This paper considers this question with the focus on dimension and primarily on inverse limits using a single set valued bonding map on one dimensional factor spaces. For example, it is known that the inverse limit with a single set valued function from an arc to an arc can have any finite dimension or even be infinite dimensional [2, Example 5, p. 129]. So we ask what sort of set valued functions yield inverse limits with dimension higher than their factor spaces.

More specifically Tom Ingram has asked if there an upper semi-continuous set valued function f from $[0,1]$ into $2^{[0,1]}$ such that the inverse limit with the single function f is homeomorphic to a 2-cell. We will show that the answer is no and we will show that there are other continua, all with dimension greater than one, that cannot be the inverse limit with a single upper semi-continuous set valued

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function from a finite graph X into 2^X. In addition it will be shown that if the upper semi-continuous set valued functions do not have a value at a point with dimension one or more, then the dimension of the inverse limit cannot be higher than the dimension of the factor spaces.

2. Definitions and Notation

All spaces are metric. A continuum is a compact and connected metric space. If $\{X_i\}$ is a countable collection of compact spaces each with metric d_i such that $diam(X_i) \leq 1$ for each i, then $\Pi_{i=1}^{\infty} X_i$ represents the countable product of the collection $\{X_i\}$, with metric given by $d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$. Note that in this article a sequence will be denoted with bold type and the terms of the sequence in italic type so that, for example, $\mathbf{x} = (x_1, x_2, x_3, \ldots)$. For each i let $\pi_i : \Pi_{i=1}^{\infty} X_i \rightarrow X_i$ be defined by $\pi_i(\mathbf{x}) = \pi_i((x_1, x_2, x_3, \ldots)) = x_i$. For each i, let $f_i : X_{i+1} \rightarrow 2^{X_i}$ be a set valued function where 2^{X_i} is the space of closed subsets of X_i with the Hausdorff metric. The inverse limit of the sequence of pairs $\{(f_i, X_i)\}$, denoted $\varprojlim(f_i, X_i)$, is defined to be the set of all $(x_1, x_2, x_3, \ldots) \in \Pi_{i=1}^{\infty} X_i$ such that $x_i \in f_i(x_{i+1})$ for each i. The functions f_i are called bonding maps, and the spaces X_i are called factor spaces. A set valued function $f : X \rightarrow 2^Y$ into the closed subsets of Y is upper semi-continuous (usc) if for each open set $V \subset Y$ the set $\{x : f(x) \subset V\}$ is an open set in X. For the countable product of a single space X define the shift map $\sigma : \Pi_{i=1}^{\infty} X \rightarrow \Pi_{i=1}^{\infty} X$ by $\sigma((x_1, x_2, x_3, \ldots)) = (x_2, x_3, x_4, \ldots)$. We say a subset $M \subseteq \Pi_{i=1}^{\infty} X$ is shift invariant if $\sigma(M) = M$. In this paper $dim(X)$ refers to the covering dimension. That is, for a compact set X, $dim(x) \leq n$ if and only if for each $\epsilon > 0$ there is an open cover of X with mesh less than ϵ and order less than or equal to n, where the order of a cover is the largest integer n such that there are $n+1$ members of the cover which have non-empty intersection [1, Corollary to Theorem V 8, p. 67].

Finally, a subcontinuum A of a continuum X is a free arc if and only if A is an arc such that the boundary of A in X is contained in the set of endpoints of A, and a continuum X is a finite graph if X is the union of a finite number of free arcs.

3. Inverse Limits with a Single Set Valued Function

If $G \subset X \times X$ and G maps onto X with both projections, then G is closed if and only if G is the graph of a usc function $f : X \rightarrow 2^X$ [2, Theorem 2.1 p. 120]. We will use $\varprojlim G$ to refer to the inverse limit of the single function from X to 2^X whose graph is G and in so doing will always imply that G maps onto X with both projections. Let $I = [0,1]$. An example is given in [2, Example 5, p. 129] of a closed subset G of $I \times I$ so that $\varprojlim G$ is homeomorphic to $I \times I \cup ([-1,0] \times \{0\})$. So it is natural to ask if there is a closed subset $G \subset I \times I$ such that $\varprojlim G$ is homeomorphic to $I \times I$. According to the theorem below the answer is no.

The proof of the following theorem depends at a crucial point on the following observations about the shift map. First, if $M = \varprojlim G$ for some closed $G \subset X \times X$, then $\sigma(M) = M$. Second, since the shift map is continuous, if K is a compact subset of M such that $\sigma|_K$ is one-to-one, then K and $\sigma(K)$ are homeomorphic subsets of M. In particular, if K is a compact subset of M such that $\pi_1(K)$ is a singleton set, then K and $\sigma(K)$ are homeomorphic subsets of M.

Lemma 3.1. *Suppose X is a compact space and $M = \varprojlim G$ for some closed set $G \subset X \times X$. If $x \in X$ such that $dim(\pi_1^{-1}(x) \cap M) > 0$, then there is a non-degenerate continuum $J \subset X$ and a set $J^* \subset \pi_1^{-1}(x) \cap M$ such that J^* is homeomorphic to $\pi_1^{-1}(J) \cap M$.*

PROOF. Assume the hypotheses are true for X, G, and M, and for convenience let $p_i = \pi_i|_M$. Suppose $x \in X$ such that $dim(p_1^{-1}(x)) > 0$. Since $p_1^{-1}(x)$ is compact, $p_1^{-1}(x)$ contains a non-degenerate continuum L [1, Proposition D, p. 22]. This continuum L must have at least one projection that is a non-degenerate continuum, though $p_1(L) = \{x\}$. So there is an $m > 1$ such that $p_m(L)$ is a non-degenerate subcontinuum J of X, and each $p_i(L)$ for $i < m$ is a singleton set $\{x_i\}$. It follows from these properties of L that if $w \in J = p_m(L)$, then $(w, x_{m-1}) \in G$. Let L^* be the set of all sequences in M of the form $(x_1, x_2, \ldots, x_{m-1}, z_1, z_2, z_3, \ldots)$ where $z_1 \in J$. Then $L \subseteq L^* \subseteq p_1^{-1}(x)$, and $p_m(L^*) = J$. Also if $(w_1, w_2, w_3, \ldots) \in p_1^{-1}(J)$ then $w_1 \in J$ so, as noted above, $(w_1, x_{m-1}) \in G$. So $(x_1, x_2, \ldots, x_{m-1}, w_1, w_2, \ldots) \in L^*$. Therefore, if $(w_1, w_2, w_3, \ldots) \in p_1^{-1}(J)$, then $(w_1, w_2, w_3, \ldots) \in \sigma^{m-1}(L^*)$. So $p_1^{-1}(J) \subseteq \sigma^{m-1}(L^*)$. But $\sigma^{m-1}(L^*)$ is homeomorphic to L^*. It follows that $p_1^{-1}(x)$ contains a set J^* homeomorphic to $p_1^{-1}(J)$. □

First we will give a simple proof that $[0,1] \times [0,1]$ cannot be homeomorphic to $\varprojlim G$ for any closed subset G of $[0,1] \times [0,1]$, thus answering Ingram's question.

Theorem 3.2. *Suppose G is a closed subset of $[0,1] \times [0,1]$ that maps onto $[0,1]$ with both projections, then $\varprojlim G$ is not homeomorphic to $[0,1] \times [0,1]$.*

PROOF. Assume G is a closed subset of $[0,1] \times [0,1]$ that maps onto $[0,1]$ with both projections, and $M = \varprojlim G$ is homeomorphic to $[0,1] \times [0,1]$. For convenience let $p_i = \pi_i|_M$.

Let $x \in (0,1)$. Then x separates $[0,1]$ so $p_1^{-1}(x)$ separates M. But M cannot be separated by a zero dimensional set since M is homeomorphic to $[0,1] \times [0,1]$ [1, Corollary I to Theorem IV 4, p. 48]. Therefore $dim(p_1^{-1}(x)) > 0$. By Lemma 3.1 there is a non-degenerate subcontinuum J of $[0,1]$ such that $p_1^{-1}(J)$ is homeomorphic to a subset of $p_1^{-1}(x)$. But J contains a nonempty open subset of $[0,1]$, so $p_1^{-1}(J)$ contains a nonempty open subset of M . Now a subset of $[0,1] \times [0,1]$ has dimension two if and only if that subset contains a nonempty open subset of $[0,1] \times [0,1]$[1, Corollary I to Theorem IV 3, p. 46]. Since M is homeomorphic to $[0,1] \times [0,1]$ it follows that $dim(p_1^{-1}(J)) = 2$. But then it also follows that $dim(p_1^{-1}(x)) = 2$, and therefore $p_1^{-1}(x)$ contains a nonempty open subset of M.

Therefore $p_1^{-1}(x)$ contains a nonempty open subset of M for each $x \in (0,1)$. So M contains an uncountable pairwise disjoint collection of nonempty open sets which contradicts the fact that M is contained in $\Pi_{i=1}^{\infty}[0,1]$ which has a countable basis. □

Now we generalize this result. For example, it follows from the theorem below that if X is a finite graph, then there is no closed subset $G \subseteq X \times X$ that maps onto X with both projections, and $n > 2$ such that $\underleftarrow{\lim} G$ is a continuum that is homeomorphic to a finite union of n-cells and n-manifolds.

Theorem 3.3. *Suppose $n > 1$, X is a continuum, G is a closed subset of $X \times X$ that maps onto X with both projections, and $M = \underleftarrow{\lim} G$ is a continuum that is homeomorphic to the union of a countable collection $\{K_i\}$ of n-cells and compact n-manifolds such that there is no uncountable collection of pairwise disjoint zero dimensional closed subsets of M each of which separates M. Then X contains a non-degenerate subcontinuum that does not contain a nonempty open set in X.*

PROOF. Assume the hypotheses are true for X, G, M, and $\{K_i\}$, and in order to get a contradiction assume every non-degenerate subcontinuum of X contains a nonempty open set. For each i, let $p_i = \pi_i|_M$ be the restriction to M of the i^{th} projection of $\Pi_{i=1}^{\infty}X$ onto X. Note that since M is a countable union of n-dimensional closed sets, M is also n-dimensional [1, Theorem III 2, p. 30].

Since every non-degenerate subcontinuum of X contains a nonempty open set, X does not contain a continuum of convergence. Therefore X is hereditarily locally connected [3, p. 167]. Thus X is arc connected [3, p. 130], and therefore X contains a non-degenerate arc, and that arc contains a free arc D. The free arc D contains an uncountable pairwise disjoint collection of pairs of points $\{a_\alpha, b_\alpha\}_{\alpha \in \Lambda}$ each of which separates X. Therefore $p_1^{-1}(\{a_\alpha, b_\alpha\})$ is a closed set that separates

M for each $\alpha \in \Lambda$. Therefore, according to the assumption that there is no uncountable pairwise disjoint collection of nonempty zero dimensional closed subsets of M each of which separates M , $dim(p_1^{-1}(\{a_\alpha, b_\alpha\})) > 0$ for uncountably many $\alpha \in \Lambda$. Since $p_1^{-1}(\{a_\alpha, b_\alpha\}) = p_1^{-1}(a_\alpha) \cup p_1^{-1}(b_\alpha)$ for each α, it follows that there exists an uncountable subset A of X such that $dim(p_1^{-1}(x)) > 0$ for each $x \in A$.

Let $x \in A$. By Lemma 3.1 there is a non-degenerate continuum $J \subset X$ such that $p_1^{-1}(x)$ contains a set homeomorphic to $p_1^{-1}(J)$. Since J has nonempty interior in X, the set $p_1^{-1}(J)$ has nonempty interior in M. According to the Baire Category Theorem there is at least one i such that $p_1^{-1}(J) \cap K_i$ has nonempty interior in K_i and K_i is homeomorphic to either an n-cell or a compact n-manifold, so $p_1^{-1}(J) \cap K_i$ has dimension n [1, Corollary I to Theorem IV 3, p. 46]. Therefore $p_1^{-1}(x)$ has dimension n. Since an n dimensional space cannot be the countable union of closed subsets with dimension less than n [1, Theorem III 2, p. 30], there is at least one j such that $p_1^{-1}(x) \cap K_j$ has dimension n. Again, since K_j is homeomorphic to either an n-cell or a compact n-manifold, $p_1^{-1}(x) \cap K_j$ has nonempty interior in K_j [1, Corollary I to Theorem IV 3, p. 46]. Since this is true for each $x \in A$, and A is uncountable, there is an l such that K_l contains an uncountable pairwise disjoint collection of sets with nonempty interior in K_l. But K_l is contained in $\Pi_{i=1}^{\infty} X$ which has a countable basis. This is a contradiction. Therefore, X contains a non-degenerate subcontinuum that does not contain a nonempty open set in X. □

4. Upper Semi-continuity, Compact Inverse Limits, and Crossovers

In [2, Theorem 2.1, p. 120] it is shown that if $f : X \to 2^Y$ is a set valued function and X and Y are compact, then f is usc if and only if the graph of f is closed. It follows that if $g : X \to Y$ is a continuous function between compact sets X and Y, and $Y = g(X)$, then $g^{-1} : Y \to 2^X$ is an usc set valued function. It is an elementary exercise to show that if X, Y, and Z are compact, and $g : X \to 2^Z$ and $f : Z \to 2^Y$ are both usc then the function $f \circ g : X \to 2^Y$ defined by $f \circ g(x) = \{y \in Y \mid \exists z \in g(x) \text{ such that } y \in f(z)\}$ is usc.

Theorem 4.1. *If $M = \varprojlim(f_i, X_i)$ where X_i is compact, $f_i : X_{i+1} \to 2^{X_i}$, and $\pi_i(M) = X_i$ for each i, then M is compact if and only if each f_i is usc.*

Proof. Assume $M = \varprojlim(f_i, X_i)$, where X_i is compact, and $f_i : X_{i+1} \to 2^{X_i}$. If each f_i is usc, then M is compact by [2, Theorem 3.2 p.121] . So suppose M is compact. Let $p_i = \pi_i|_M$. It will be shown that $f_i = p_i \circ p_{i+1}^{-1}$ for each i.

Suppose there is an $x_i \in X_i$, and an $x_{i+1} \in X_{i+1}$ such that $x_i \in f_i(x_{i+1})$. Since f_j is defined on all of X_{j+1} for each j, there is a finite sequence $\{x_1, x_2, \ldots x_{i-1}\}$ such that $x_{i-1} \in f_{i-1}(x_i)$ and $x_j \in f_j(x_{j+1})$ for $1 \leq j \leq i-2$. Also since $p_{i+1}(M) = X_{i+1}$, there is a $\mathbf{z} \in M$ such that $p_{i+1}(\mathbf{z}) = x_{i+1}$. For each $j > i+1$ let $x_j = p_j(\mathbf{z})$. Let $\mathbf{w} = (x_1, x_2, \ldots, x_i, x_{i+1} \ldots)$. Then $\mathbf{w} \in M$, $p_i(\mathbf{w}) = x_i$, and $p_{i+1}(\mathbf{w}) = x_{i+1}$. It follows that $x_i \in p_i \circ p_{i+1}^{-1}(x_{i+1})$.

Now suppose there is an $x_i \in X_i$ and an $x_{i+1} \in X_{i+1}$ such that $x_i \in p_i \circ p_{i+1}^{-1}(x_{i+1})$. Then there is a $\mathbf{w}$ in $M = \underleftarrow{\lim}(f_i, X_i)$ such that $p_{i+1}(\mathbf{w}) = x_{i+1}$ and $p_i(\mathbf{w}) = x_i$. From the definition of $\underleftarrow{\lim}(f_i, X_i)$ it follows that $x_i \in f_i(x_{i+1})$.

It has been shown that $f_i = p_i \circ p_{i+1}^{-1}$. Since each f_i is the composition of usc functions, each f_i is usc. $\square$

Suppose $\{X_i\}$ is a sequence of spaces and $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \Pi_{i=1}^{\infty} X_i$ and $\mathbf{y} = (y_1, y_2, y_3, \ldots) \in \Pi_{i=1}^{\infty} X_i$ and $x_i = y_i$ for some i, then define $Cr_i(\mathbf{x}, \mathbf{y}) = (x_1, x_2, \ldots, x_i, y_{i+1}, y_{i+2}, \ldots)$. Note that $Cr_i(\mathbf{x},\mathbf{x}) = \mathbf{x}$ for each $\mathbf{x} \in \Pi_{i=1}^{\infty} X_i$ and each i. For a subset M of $\Pi_{i=1}^{\infty} X_i$, let $Cr(M)$ be the set of all $\mathbf{z} \in \Pi_{i=1}^{\infty} X_i$ such that there are an i , $\mathbf{x} \in M$, and $\mathbf{y} \in M$ such that $\mathbf{z} = Cr_i(\mathbf{x}, \mathbf{y})$. Note that $M \subseteq Cr(M)$. We say a subset $M \subseteq \Pi_{i=1}^{\infty} X_i$ contains all crossovers if $Cr(M) = M$. It is easy to verify that if there are functions $f_i : X_{i+1} \to 2^{X_i}$, and $M = \underleftarrow{\lim}(f_i, X_i)$, then $Cr(M) = M$.

Theorem 4.2. *Suppose each X_i is a compact space, and M is a compact subset of $\Pi_{i=1}^{\infty} X_i$, and $X_i' = \pi_i(M)$ for each i, then the following are equivalent:*

i. *There exist set valued functions $g_i : X_{i+1}' \to 2^{X_i'}$ such that $M = \underleftarrow{\lim}(g_i, X_i')$.*
ii. *There exist usc set valued functions $f_i : X_{i+1}' \to 2^{X_i'}$ such that $M = \underleftarrow{\lim}(f_i, X_i')$.*
iii. *M contains all crossovers.*

PROOF. Assume each X_i is a compact space and M is a compact subset of $\Pi_{i=1}^{\infty} X_i$, and $X_i' = \pi_i(M)$ for each i. That property ii follows from property i is established in the previous theorem. It is easy to see that an inverse limit contains all crossovers, so property ii implies property iii. So all that remains is to show that if M contains all crossovers, then there exist set valued functions $g_i : X_{i+1}' \to 2^{X_i'}$ such that $M = \underleftarrow{\lim}(g_i, X_i')$.

Assume M contains all crossovers. Let $p_i = \pi_i|_M$. Define the set valued function $g_i : X_{i+1}' \to 2^{X_i'}$ by $g_i = p_i \circ p_{i+1}^{-1}$. If $(x_1, x_2, x_3, \ldots) \in M$, then $x_i \in p_i \circ p_{i+1}^{-1}(x_{i+1}) = g_i(x_{i+1})$ for each i. So $M \subseteq \underleftarrow{\lim}(g_i, X_i')$.

Now assume $(x_1, x_2, x_3, \ldots) \in \varprojlim(g_i, X_i')$. For each i, since $x_i \in g_i(x_{i+1}) = p_i \circ p_{i+1}^{-1}(x_{i+1})$, it follows that there is a $\mathbf{z}^i \in p_{i+1}^{-1}(x_{i+1})$ such that $p_i(\mathbf{z}^i) = x_i$. That is, there is a $\mathbf{z}^i \in M$ such that $p_i(\mathbf{z}^i) = x_i$ and $p_{i+1}(\mathbf{z}^i) = x_{i+1}$. Now construct inductively the sequence $\{\mathbf{w}^n\}$ in M with $p_i(\mathbf{w}^n) = x_i$ for $i \leq n$. Let $\mathbf{w}^1 = \mathbf{w}^2 = \mathbf{z}^1$. Then $p_i(\mathbf{w}^2) = x_i$ for $i \leq 2$. Assume there is a $\mathbf{w}^k \in M$ such that $p_i(\mathbf{w}^k) = x_i$ for $i \leq k$. Then, since M contains all crossovers, M contains $\mathbf{w}^{k+1} = Cr_k(\mathbf{w}^k, \mathbf{z}^k)$. It is easy to verify that $p_i(\mathbf{w}^{k+1}) = x_i$ for $i \leq k+1$. Therefore, for each n there is a $\mathbf{w}^n \in M$ such that $p_i(\mathbf{w}^n) = x_i$ for each $i \leq n$. The sequence $\{\mathbf{w}^n\}$ converges to $(x_1, x_2, x_3, \ldots)$, and, since M is closed, $(x_1, x_2, x_3, \ldots) \in M$. □

From Theorem 4.2 there follows a different way to formulate questions. For if X is a compact space, and $M \subset \Pi_{i=1}^{\infty} X$ such that $\pi_i(M) = X$ for each i, $Cr(M) = M$, and $\sigma(M) = M$, and if $G = \{(x, y) \in X \times X \mid y \in \pi_1(\pi_2^{-1}(x) \cap M))\}$, then $M = \varprojlim G$. Therefore the question of Ingram is equivalent to asking if the Hilbert cube contains a subset K homeomorphic to $[0,1] \times [0,1]$ such that $\sigma(K) = K$, $Cr(K) = K$, and $\pi_i(K) = [0,1]$ for each i . In the case of the Hilbert Cube the question is the same without the condition that $\pi_i(K) = [0,1]$ since it follows from the shift invariance that all of the projections are equal to the same interval, and the infinite product of that interval is homeomorphic to the Hilbert cube. In the previous section we showed that there are many different continua, all having dimension at least two, including $[0,1] \times [0,1]$, that are not homeomorphic to a subset K of the Hilbert cube such that $\sigma(K) = K$ and $Cr(K) = K$.

Question. Is there a one dimensional continuum that is not homeomorphic to a subset K of the Hilbert cube such that $\sigma(K) = K$ and $Cr(K) = K$.

5. Inverse Limits with Dimension Higher than the Dimension of the Factor Spaces

As noted above the inverse limit with the interval $[0,1]$ as the factor space and usc set valued functions can have dimension two, and in fact it can have any finite dimension [5] or be infinite dimensional [4]. However, it follows from the theorem below that if the dimension of $f_i(x)$ is zero for each i and for each $x \in [0,1]$, then $\varprojlim(f_i, [0,1])$ has dimension no greater than one. So, for example, if $G \subset [0,1] \times [0,1]$ such that both projections of G are $[0,1]$ and G is the union of the graphs of finitely many continuous functions from $[0,1]$ to $[0,1]$, then $\varprojlim G$ is one dimensional.

For the next two lemmas let $\{X_i\}$ be a collection of compact spaces with $dim(X_i) \leq m$ for each i. Suppose $M = \underleftarrow{\lim}(f_i, X_i)$ where each $f_i : X_{i+1} \to 2^{X_i}$ is an usc set valued function whose graph projects onto X_{i+1} and X_i. For each $n > 1$ let $P_j = X_1 \times X_2 \times \ldots \times X_j$ with the usual product topology, and define $F_n : X_{n+1} \to 2^{P_n}$ by

$$F_n(x) = \{(x_1, x_2, \ldots, x_n) \in P_n \,|\, x_n \in f_n(x) \,\mathrm{and}\, x_i \in f_i(x_{i+1}) \,\mathrm{for}\, 1 \leq \mathrm{i} \leq \mathrm{n} - 1\}$$

Lemma 5.1. *F_n is usc for each n.*

PROOF. According to [2, Theorem 2.1, p.120] if the graph of F_n is closed then F_n is usc. Let G be the set of all sequences $(x_1, x_2, \ldots, x_n, x_{n+1}) \in P_{n+1}$ such that $x_i \in f_i(x_{i+1})$ for $i = 1, 2, \ldots n$ and $x_{n+1} \in X_{n+1}$. Then G is homeomorphic to the graph of F_n. Suppose $(z_1, z_2, \ldots, z_n, z_{n+1}) \in P_{n+1} \setminus G$. Then $z_i \in X_i \setminus f_i(X_{i+1})$ for some $i < n + 1$. Thus (z_i, z_{i+1}) is not in $G_i = \{(y, x) \,|\, y \in f_i(x)\}$ which is a closed subset of $X_i \times X_{i+1}$ since G_i is homeomorphic to the graph of f_i, and f_i is usc [2, Theorem 2.1, p.120]. So there exist open sets $V \subset X_i$ and $U \subset X_{i+1}$ such that $(z_i, z_{i+1}) \in V \times U \subseteq (X_i \times X_{i+1}) \setminus G_i$. Then $(z_1, z_2, \ldots, z_n, z_{n+1}) \in \pi_i^{-1}(V) \cap \pi_{i+1}^{-1}(U)$, $\pi_i^{-1}(V) \cap \pi_{i+1}^{-1}(U)$ is open in P_n, and $\pi_i^{-1}(V) \cap \pi_{i+1}^{-1}(U) \subseteq P_{n+1} \setminus G$. So G is closed, and therefore F_n is usc. □

Lemma 5.2. *If $dim(F_n(x)) > 0$ for some $n > 1$ and $x \in X_{n+1}$, then there is an $i \leq n + 1$ and $z \in X_{i+1}$ such that $dim(f_i(z)) > 0$.*

PROOF. Suppose $dim(F_n(x)) > 0$ for some $n > 1$ and $x \in X_{n+1}$. Since $F_n(x)$ is compact, $F_n(x)$ must contain a non-degenerate continuum K [1, Proposition D, p. 22]. Let j be the largest integer less than $n + 1$ such that $dim(\pi_j(K)) > 0$. If $j = n$, then $\pi_j(K) \subseteq f_n(x)$. So $dim(f_n(x)) > 0$. If $j < n$, then $\pi_{j+1}(K)$ is zero dimensional and connected. So $\pi_{j+1}(K) = \{z\}$ for some $z \in X_{j+1}$, and $\pi_j(K) \subseteq f_j(z)$. So $dim(f_j(z)) > 0$. □

Theorem 5.3. *If m is a positive integer, and $M = \underleftarrow{\lim}(f_i, X_i)$ where each X_i is a compact space such that $\pi_i(M) = X_i$ and $dim(X_i) \leq m$, and each f_i is an usc set valued function such that $dim(f_i(x)) = 0$ for each $x \in X_{i+1}$, then $dim(M) \leq m$.*

PROOF. Assume the hypotheses are true for M and the functions $f_i : X_{i+1} \to 2^{X_i}$. Also assume $diam(X_i) \leq 1$ for each i. Let $\epsilon > 0$, and let n be such that $\sum\limits_{i=n+1}^{\infty} \frac{1}{2^i} < \frac{\epsilon}{2}$, and let $\delta > 0$ be such that $\sum\limits_{i=1}^{n} \frac{\delta}{2^i} < \frac{\epsilon}{2}$. By Lemma 5.2 $dim(F_{n-1}(x)) = 0$ for each $x \in X_n$. For each $x \in X_n$, let $\mathcal{V}_x$ be a finite pairwise disjoint open cover of $F_{n-1}(x)$ such that for each $v \in \mathcal{V}_x$ the set of values of the i^{th} coordinate of points of v has diameter less than δ for each $i < n$. Since F_{n-1} is usc, each $x \in X_n$

is contained in an open set U_x such that $F_{n-1}(U_x) \subseteq \bigcup \mathcal{V}_x$. Since $dim(X_n) \leq m$, and X_n is compact, there is a finite open refinement $\mathcal{U}$ of $\{U_x\}_{x \in X_n}$ with mesh less than δ and order less than $m+1$ covering X_n.

For each $x \in X_n$ and each $v \in \mathcal{V}_x$ let v^* be the set of all $(x_1, x_2, x_3, \ldots) \in M$ such that $(x_1, x_2, \ldots, x_{n-1}) \in v$, and let $\mathcal{V}_x^* = \{v^* \mid v \in \mathcal{V}_x\}$. Note that if p_i is the projection of P_{n-1} onto X_i, then $v^* = \bigcap_{i=1}^{n-1} \pi_i^{-1}(p_i(v)) \cap M$, and since projection maps from product spaces with the usual product topologies are open and continuous, v^* is open in M for each $v \in \mathcal{V}_x$. Therefore $\mathcal{V}_x^*$ is a pairwise disjoint collection of open sets in M. Also note that $diam(\pi_i(v^*)) < \delta$ for each $v^* \in \mathcal{V}_x^*$ and each $i < n$. So $diam(v^*) < \epsilon$ for each $v^* \in \mathcal{V}_x^*$.

For each element $u \in \mathcal{U}$ there is an $x \in X_n$ such that $u \subseteq U_x$, and therefore $F_{n-1}(u) \subseteq \bigcup \mathcal{V}_x$. It follows that $\pi_n^{-1}(u) \subseteq \bigcup \mathcal{V}_x^*$. So the collection $\{v^* \cap \pi_n^{-1}(u) \mid v^* \in \mathcal{V}_x^*\}$ is a partition of $\pi_n^{-1}(u)$ into pairwise disjoint open sets with diameter less than ϵ. Since the order of the cover $\{\pi_n^{-1}(u) \mid u \in \mathcal{U}\}$ of M is the same as the order of $\mathcal{U}$, there is an open cover of M with order less than $m+1$ and mesh less than ϵ. □

Theorem 5.4. *If X_1 is a continuum such that every non-degenerate subcontinuum K of X_1 contains a countable set that separates K, and for each i, X_i is compact, and $f_i : X_{i+1} \to 2^{X_i}$ is usc, and for each $y \in X_i$, $dim(\{x \in X_{i+1} \mid y \in f_i(x)\}) = 0$, then $dim(\varprojlim(f_i, X_i)) \leq 1$.*

Proof. Suppose X_1 is a continuum such that every non-degenerate subcontinuum K of X_1 contains a countable set that separates K, and for each i, X_i is compact, and $f_i : X_{i+1} \to 2^{X_i}$ is usc, and for each $y \in X_i$, $dim(\{x \in X_{i+1} \mid y \in f_i(x)\}) = 0$. Let $M = \varprojlim(f_i, X_i)$.

If $z \in X_1$ and $dim(\pi_1^{-1}(z)) > 0$, then $\pi_1^{-1}(z)$ contains a non-degenerate continuum L. Let m be the smallest integer such that $dim(\pi_m(L)) > 0$. Let y be an element of X_{m-1} such that $\{y\} = \pi_{m-1}(L)$. Then $\pi_m(L) \subset \{x \in X_m \mid y \in f_{m-1}(x)\}$. This contradicts the assumption that $\{x \in X_m \mid y \in f_{m-1}(x)\}$ is zero dimensional. So $dim(\pi_1^{-1}(z)) = 0$ for each $z \in X_1$.

If $dim(M) > 1$, then, since M is compact, M contains a continuum K such that every subset of K that separates K has dimension at least one [1, Theorem VI 8, p. 94]. Now $dim(\pi_1^{-1}(x)) = 0$ for each $x \in X_1$. So $\pi_1(K)$ is a non-degenerate subcontinuum of X_1, and therefore $\pi_1(K)$ contains a countable set A that separates $\pi_1(K)$. Therefore $\pi_1^{-1}(A)$ separates K. But $\pi_1^{-1}(A)$ is the countable union of compact zero dimensional sets. So $\pi_1^{-1}(A)$ is zero dimensional. This is a contradiction. Therefore $dim(M) \leq 1$. □

A vertical line in $[0,1] \times [0,1]$ is a set of the form $\{(a, y) \mid b \le y \le c\}$ where $0 \le a \le 1$ and $0 \le b < c \le 1$, and a horizontal line is a set of the form $\{(x, a) \mid b \le x \le c\}$ where $0 \le a \le 1$ and $0 \le b < c \le 1$. The following theorem follows immediately from the previous two theorems.

Theorem 5.5. *Suppose G is a closed subset of $[0,1] \times [0,1]$. If G does not contain both a vertical line and a horizontal line then $dim(\underleftarrow{lim} G) \le 1$.*

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