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Octary Codewords with Power Envelopes of $3*2^m$

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Abstract

This paper examines codewords of length 2^m in Z_8 with envelope power maxima of $3 * 2^m$. Using the general form for Golay pairs as a base, a general form is derived for the set of coset leaders that generate these codewords. From this general form it will be proven that there exists at least one element in the coset that achieves a power of $3 * 2^m$ for each m-even and m-odd case.

1 Introduction

1.1 Purpose

This paper examines the envelope powers of codewords in Z_8, specifically those that reach an integer maxima of $3 * 2^m$. The envelope power describes the upper bound of the signaling power used in transmitting the codeword. Codewords with low envelope power are more desirable for efficiency in engineering purposes. When transmitting many codewords at one time, low envelope power of individual codewords contribute to a lower overall power. A special set of codewords known as Golay pairs are understood to have good power properties with an upper bound of 2^{m+1}. Similarly, there exist Golay quadruples that reach power maxima of 2^{m+2}. The codewords this paper examines have power maxima in between the Golay pairs and Golay quadruples. These codewords have not been studied to the same extent as the Golay codewords. It is important to understand these additional codewords with good power properties in order to transmit more information efficiently.

1.2 Background

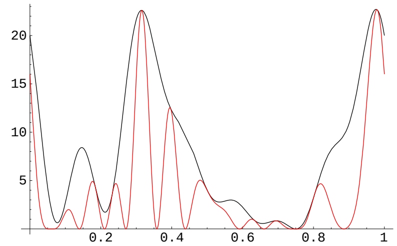
1.2.1 Signal/Power Formulas

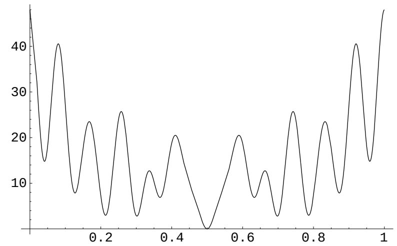
The envelope power of a codeword is the power required to transmit the signal. The power of an individual codeword is calculated using the complex signal, which is a composition of phase shifts. There is a particular frequency associated with each position. The value of each position is encoded as a phase shift in the oscillating wave with frequency corresponding to its position. In octary the eight possible values are mapped onto the eighth roots of unity using $(e^{2\pi i/8})^j$ where j is the original octary value. The values from 0 to 7 are as follows: 1, $1/\sqrt{2}+i/\sqrt{2}$, i, $-1/\sqrt{2}+i/\sqrt{2}$, -1, $-1/\sqrt{2}-i/\sqrt{2}$, $-i$, $1/\sqrt{2}-i/\sqrt{2}$. The values are encoded into phase shifts using the formula:

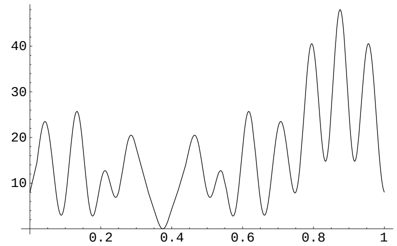
$$d_n e^{i2\pi ft}$$

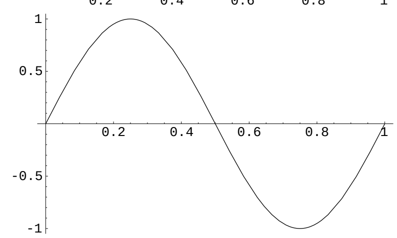
where d_n is the value at position n in the codeword, t is the time (from 0 to 1), and f is the frequency associated with position n.

For example, each position of the codeword 6 0 1 7 is encoded as follows, using the position n as the frequency f. In actual transmission the signal sent is the real part of the complex signal.







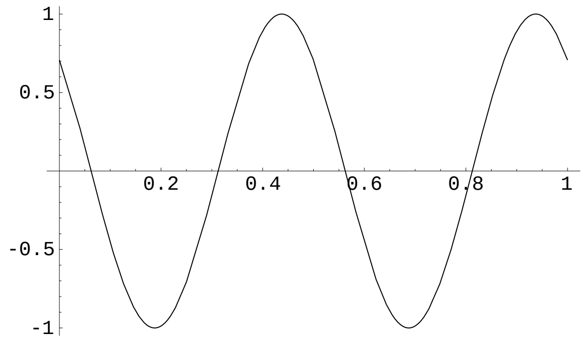


These phase shifts are transmitted simultaneously rather than sequentially. In this way more information can be sent at one time because we can let the symbol period be longer and there is less opportunity for interference. When transmitting through a medium such as air, interference would disrupt a long sequential signal. The complex signal for the codeword, $S(t)$, is the sum of the individual waves for each position:

$$S(t) = \sum_{n=1}^{N} d_n e^{i2\pi(f_c+f_n)t}$$

where N is the length (2^m), f_c is the carrier frequency for the signal, and f_n is the frequency offset used to encode position n.

The signal for the previous example with the codeword 6 0 1 7 is shown below.



The envelope power, $P(t)$, is defined as:

$$
\begin{aligned}
P(t) &= S(t)S^*(t) \\
&= \sum_{n=1}^{N} d_n e^{i2\pi(f_c+f_n)t} \sum_{m=1}^{N} d_m^* e^{-i2\pi(f_c+f_m)t} \\
&= \sum_{n,m} d_n d_m^* e^{i2\pi(f_n-f_m)t}
\end{aligned}
$$

The carrier frequency f_c cancels out of the equation. Therefore the envelope power of any signal is independent of its frequency.

The frequency, f_n, associated with position, n, can be written as nf_s. In the same way, f_m is written as mf_s. The power formula then becomes:

$$
\begin{aligned}
P(t) &= \sum_{n,m} d_n d_m^* e^{i2\pi(n-m)f_s t} \\
&= N + \sum_{n\neq m} d_n d_m^* e^{i2\pi(n-m)f_s t} \\
&= N + \sum_{u>0}(\sum_{n|1\leq n\leq N-u} d_n d_{n+u}^*)e^{i2\pi u f_s t} + \sum_{u<0}(\sum_{n|-u+1\leq n\leq N} d_n d_{n+u}^*)e^{i2\pi u f_s t} \\
&= N + 2Re(\sum_{u>0} C_D(u)e^{i2\pi u f_s t})
\end{aligned}
$$

$C_D(u)$ is known as the aperiodic auto correlation of the codeword D.

$$C_D(u) = \sum_{n|1\leq n\leq N-u} d_n d_{n+u}^*$$

For a Golay pair (A and B), $C_A(u) + C_B(u) = 0$ for all $u > 0$. This gives an upperbounds on the power function for a Golay codeword as seen below.

$$P_A(t) = N + 2Re(\sum_{u>0} C_A(u)e^{i2\pi u f_s t})$$

$$
\begin{aligned}
P_B(t) &= N + 2Re(\sum_{u>0} C_B(u)e^{i2\pi u f_s t}) \\
P_A(t) &+ P_B(t) = 2N + 2Re(\sum_{u>0}(C_A(u) + C_B(u))e^{i2\pi u f_s t}) = 2N
\end{aligned}
$$

A bound of 2N is much better in comparison to the worst case scenario seen with the all 0 codeword. All the peaks would line up at t=0, resulting in a power of N^2.

Using the property that $e^{ix} = \cos x - i \sin x$, we can rewrite the equation for the complex signal $S(t)$ as:

$$
S(t) = \sum_{n=1}^{N} d_n \cos(2\pi nt) + i \sum_{n=1}^{N} d_n \sin(2\pi nt)
$$

Calculations with this equation are less complicated and therefore it will be used instead of the former equation.

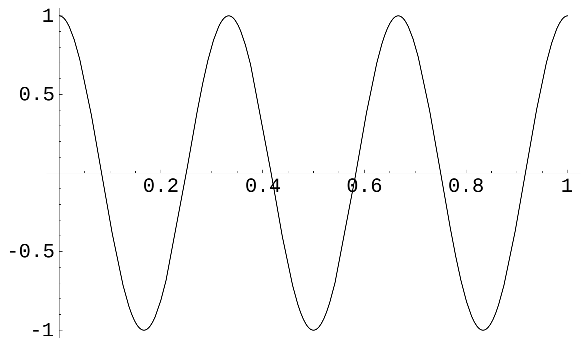
For example, consider the codeword 0 0 0 2 0 4 2 0 at $t = 0$. $S(0)$ simplifies to:

$$
\begin{aligned}
S(0) &= \sum_{n=1}^{N} d_n \\
&= 1 + 1 + 1 + i + 1 + (-1) + i + 1 \\
&= 4 + 2i
\end{aligned}
$$

$P(0)$ is easily calculated:

$$
\begin{aligned}
P(0) &= S(0)S^*(0) \\
&= (4 + 2i)(4 - 2i) \\
&= 20
\end{aligned}
$$

In the graph below, the lighter line is the signal for this codeword and the dark line is the power envelope.



For information consult [Davis/Jedwab].

1.2.2 Reed-Muller Codes

Reed-Muller Codes are linear codes with good error correcting capabilities. The notation used is $RM(r,m)$. This describes the Reed-Muller Code of order r and of length 2^m. First order Reed-Muller Codes are made up of $m+1$ basis vectors and all linear combinations of those vectors. The basis vectors are as follows:

$\mathbf{1}$	The all one codeword.
x_1	2^{m-1} 0's followed by 2^{m-1} 1's
x_i	2^{m-i} 0's followed by 2^{m-i} 1's repeated i times.

Second order Reed-Muller Codes include all first order codewords and the intersections of all the first order codewords. For example, the generator matrix for $RM(2,3)$ is shown below:

$$
\begin{array}{llllllll l}
\mathbf{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
x_1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
x_2 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
x_3 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
x_1x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
x_1x_3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
x_2x_3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}
$$

This paper examines sets of codewords called cosets. The coset is described using Reed-Muller codes. A coset of 1st order Reed-Muller codes is defined as: $x + RM(1,m) = x + c | c \in RM(1,m)$. The x term is known as the coset leader and c is called the offset. The coset leaders studied in this paper are of the form: $2\sum_{i<j} u_{ij}x_ix_j$. Over Z_8 there are 8^m choices of offsets, so there are 8^m elements in a given coset.

Second order RM codes are used to construct the Golay pairs mentioned earlier. A Golay coset leader has the form:

$$x_{\alpha_1}x_{\alpha_2} + x_{\alpha_2}x_{\alpha_3} + \cdots + x_{\alpha_{m-1}}x_{\alpha_m}$$

One property of Reed-Muller codes that is extremely useful in proving general cases is the weight equivalence of codewords. The weight of a codeword is the number of non-zero entries. In binary the weight would be the number of 1's. The codeword 0 0 1 0 has a weight of 1. All linear combinations of any n first order codes in binary will have the same weight.
For example, in $RM(1,3)$:
$x_1 + x_2 =$ 0 0 1 1 1 1 0 0, weight is 4
$x_2 + x_3 =$ 0 1 1 0 0 1 1 0, weight is 4
The same occurs for any order of Reed-Muller codes and combinations thereof. In octary, not only is weight conserved but all entry values are conserved.
For example, in $RM(2,3)$:

$2x_1$	0 0 0 0 2 2 2 2	$2x_2$	0 0 2 2 0 0 2 2
x_3	0 1 0 1 0 1 0 1	x_3	0 1 0 1 0 1 0 1
$3x_1x_3$	$\underline{0\ 0\ 0\ 0\ 0\ 3\ 0\ 3}$	$3x_2x_3$	$\underline{0\ 0\ 0\ 3\ 0\ 0\ 0\ 3}$
	0 1 0 1 2 6 2 6		0 1 2 6 0 1 2 6

By permutating the subscripts of the basis vectors, the same positions in each codeword exchange values. This only affects the positions of the entry values, not the values themselves. In the example, when changing x_1 to x_2, the third and fifth positions and the fourth and sixth positions exchange values in all the terms and the final sum.

For more informatin on Reed-Muller codes consult [MacWilliams/Sloane]

1.2.3 Proof Techniques

The Reed-Muller Codes are used in constructing important codewords such as the Golay pairs and Golay quadruples. Another unique property of Reed-Muller vectors is that they can be used to construct other Reed-Muller vectors. Specifically, Reed-Muller vectors of length 2^m are constructed by concatenating Reed-Muller basis vectors of length 2^{m-1}. There are several concatenation lemmas listed below. Let $\mathbf{0}$ represent the all-0 codeword and $\mathbf{1}$ represent the all-1 codeword. The $|$ symbol represents a concatenation.

Concatenation Lemmas:

(i) $[\mathbf{1} \mid \mathbf{1}]_{m-1} = [\mathbf{1}]_m$

(ii) $[\mathbf{0} \mid \mathbf{1}]_{m-1} = [x_1]_m$

(iii) $[\mathbf{0} \mid x_a]_{m-1} = [x_1 x_{a+1}]_m$

(iv) $[x_a \mid x_a]_{m-1} = [x_{a+1}]_m$

(v) $[x_{a-1}x_{b-1} \mid x_{a-1}x_{b-1}]_{m-1} = [x_a x_b]_m$

(vi) All of these lemmas are both additive and multiplicative.

For an example of Concatenation Lemma (ii), consider x_1 of length 2^3. x_1 is defined as 2^{m-1} 0's concatenated with 2^{m-1} 1's. For this example, there will be 2^2 0's followed by 2^2 1's.

$$\begin{aligned} 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1 &= [0\ 0\ 0\ 0 \mid 1\ 1\ 1\ 1] \\ &= [0 \mid 1]_2 = [x_1]_3 \\ &= [0 \mid 1]_{m-1} = [x_1]_m \end{aligned}$$

For an example of Concatenation Lemma (iv), consider x_2 of length 2^3. x_2 is defined as 2^{m-2} 0's concatenated with 2^{m-2} 1's repeated twice. For this example, there will be 2 0's followed by 2 1's repeated twice.

$$\begin{aligned} 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1 &= [0\ 0\ 1\ 1 \mid 0\ 0\ 1\ 1] \\ &= [x_1 \mid x_1]_2 = [x_2]_3 \\ &= [x_1 \mid x_1]_{m-1} = [x_2]_m \end{aligned}$$

An example of Lemma (vi) is shown below.

$$\begin{aligned} [x_a + x_b \mid x_a + x_b]_{m-1} &= [x_a \mid x_a]_{m-1} + [x_b \mid x_b]_{m-1} \\ &= [x_{a+1} + x_{b+1}]_m \end{aligned}$$

This simplification is done using the lemma $[x_a \mid x_a]_{m-1} = [x_{a+1}]_m$ and the additive properties of the binary vectors.

These concatenation lemmas are used to decompose codewords and prove theorems dealing with the weights of codewords. The weight of a codeword in binary (Z_2) is simply the number of 1s in the codeword. The weight is a useful property because it can tell us the number of 0s and 1s in a binary codeword. By breaking down octary codewords into binary components, the number of 0's, 1's, 2's, ..., 7's can be determined. This information can be used to calculate the power of the codeword. Consider the previous example with the codeword 0 0 0 2 0 4 2 0. This codeword is the sum of second order Reed-Muller codes, specifically $4x_1x_3 + 2x_2x_3 + 2x_1x_2$. The concatenation lemmas are used to decompose the codeword to a simpler form where it would be obvious that the codeword contained five 0s, two 2s, and one 4. Then:

$$\begin{aligned} S(0) &= 5(1) + 2(i) + 1(-1) \\ &= 4 + 2i \\ P(0) &= 20 \end{aligned}$$

There are theorems on the weights of Golay pairs that are very useful. Let L_m represent the Golay coset leader $x_1x_2 + x_2x_3 + \cdots + x_{m-1}x_m$ of length 2^m. Let α be a binary value (0 or 1).

Theorem 1 *For codewords of length 2^m with m-even, the codeword of form $[L_m + \alpha x_1]_m$ has weight $2^{m-1} - 2^{\frac{m}{2}-1}$.*

Theorem 2 *For codewords of length 2^m with m-odd, the codeword L_m has weight $2^{m-1} - 2^{\frac{m-1}{2}}$.*

Although these weights are associated with Golay pairs, they will be essential in proving the existence of a codeword that reaches power maxima of $3 * 2^m$. These two theorems and the concatenation lemmas will be used extensively in the proofs in the third section. For proofs of these theorems, consult [Cammarano, Walker].

The following two lemmas provide some insight into why using an octary base will produce a factor of 3 in the power. As stated earlier, the codeword values are encoded to be used in the signal formula using the mapping:$j \rightarrow (e^{2\pi i/8})^j$. The result of the signal formula is then multiplied by its complex conjugate to obtain the power.

Lemma 3 *The sum, $(e^{2\pi i/8})^x + (e^{2\pi i/8})^{x+1} + (e^{2\pi i/8})^{x+3}$, multiplied by its complex conjugate gives 3.*

Proof:

$[(e^{2\pi i/8})^x + (e^{2\pi i/8})^{x+1} + (e^{2\pi i/8})^{x+3}][(e^{2\pi i/8})^{-x} + (e^{2\pi i/8})^{-x-1} + (e^{2\pi i/8})^{-x-3}]$
$= (e^{2\pi i/8})^x[1 + e^{2\pi i/8} + (e^{2\pi i/8})^3](e^{2\pi i/8})^{-x}[1 + e^{-2\pi i/8} + (e^{2\pi i/8})^{-3}]$
$= (1 + i\sqrt{2})(1 - i\sqrt{2}) = 3$

□

Lemma 4 *The sum,* $2(e^{2\pi i/8})^x + (e^{2\pi i/8})^{x+1} + (e^{2\pi i/8})^{x+3}$*, multiplied by its complex conjugate gives 6.*

Proof:
$[2(e^{2\pi i/8})^x + (e^{2\pi i/8})^{x+1} + (e^{2\pi i/8})^{x+3}][2(e^{2\pi i/8})^{-x} + (e^{2\pi i/8})^{-x-1} + (e^{2\pi i/8})^{-x-3}]$
$= (e^{2\pi i/8})^x[2 + e^{2\pi i/8} + (e^{2\pi i/8})^3](e^{2\pi i/8})^{-x}[2 + e^{-2\pi i/8} + (e^{2\pi i/8})^{-3}]$
$= 2(\sqrt{2} + i)(\sqrt{2} - i) = 6$

□

Note that x, $x-1$, and $x-3$ can also be used. This can be seen by switching $-x$ with x. In future computations, signals with sums $2^a((e^{2\pi i/8})^x + (e^{2\pi i/8})^{x+1} + (e^{2\pi i/8})^{x+3})$ will be used.

2 Observations

2.1 Codewords of length 8 and 16

2.1.1 Coset Leaders

Upon examination of the length 8 and length 16 codewords that reached a max of 3N (N = 2^m), a connection was found between these codewords and the Golay codewords. The 3N-coset leaders can be derived from the Golay coset leaders. Length 8 Golay coset leaders have the form:

$$4x_ax_b + 4x_bx_c$$

By adding or subtracting twice the second term and adding 6 or 2 times x_ax_c to a Golay coset leader, a 3N-coset leader is generated.

$$4x_ax_b + 4x_bx_c \pm 2x_bx_c \pm 2x_ax_c$$

Length 16 Golay coset leaders are of the form:

$$4x_ax_b + 4x_bx_c + 4x_cx_d$$

Again by adding or subtracting twice the second term and adding 6 or 2 times x_ax_c to a Golay coset leader, a 3N-coset leader is generated.

$$4x_ax_b + 4x_bx_c + 4x_cx_d \pm 2x_bx_c \pm 2x_ax_c$$

These formulas each generate four different forms for the 3N-coset leaders. Counting the permutations, there would be $4 * 3!$ 3N-coset leaders for length 8 and $4 * 4!$ for length 16. This however does not account for the overlap in permutations. Switching the a and b in the different forms generates the same coset leader.

$$4x_ax_b + 4x_bx_c + 4x_cx_d - 2x_bx_c + 2x_ax_c = 4x_bx_a + 4x_ax_c + 4x_cx_d - 2x_ax_c + 2x_bx_c$$

$$4x_ax_b + 4x_bx_c + 4x_cx_d + 2x_bx_c - 2x_ax_c = 4x_bx_a + 4x_ax_c + 4x_cx_d + 2x_ax_c - 2x_bx_c$$

$$4x_ax_b + 4x_bx_c + 4x_cx_d - 2x_bx_c - 2x_ax_c = 4x_bx_a + 4x_ax_c + 4x_cx_d + 2x_ax_c + 2x_bx_c$$

Therefore dividing by 2 will yield the correct number of 3N-coset leaders.

2.1.2 Within a Coset

For length 8, each coset contained exactly 16 elements that hit a power max of 24. (This excludes the linear combinations containing the all 1 codeword.) For length 16, there were exactly 32 elements in the coset with a max of 48. All other elements had maximum powers below these values. Example cosets for both 8, 16, and 32 lengths are listed in Appendix B, C, and D respectively.

All occurrences of the 3N max occur at positions where $t = n/8$ (where n is an integer from 1–7). Each multiple of 1/8 occurs as the max position for exactly two of the 16 elements of length 8 and four of the 32 elements of length 16. Upon examination of the graphs of the power functions for these codewords, there were two sets of graphs within each length 8 coset and 4 within each length 16 coset. There were eight graphs in each set that were all phase shifts of the same function. The following theorem proves that the phase shift is 1/8.

Theorem 5 *Adding $4x_{m-2}+2x_{m-1}+x_m$ to a codeword will shift the power function to the left by 1/8.*

Proof: Given a codeword d of length N where $d =$ a b c d e f g h $\ldots \alpha_N$:

$$S(t) = ae^{2\pi it} + be^{4\pi it} + ce^{6\pi it} + de^{8\pi it} + \cdots$$

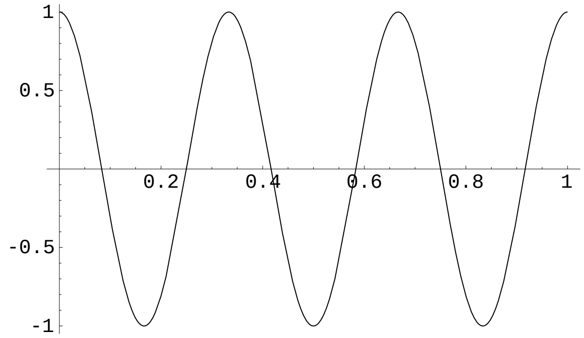
Adding $4x_{m-2}+2x_{m-1}+x_m$ (the codeword 0 1 2 3 4 5 6 7 0 1 2 3 $\ldots$) gives a new codeword d'.

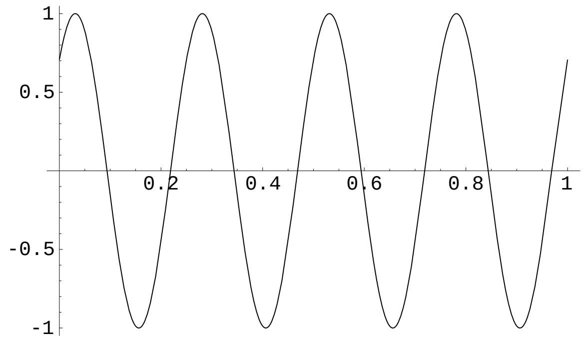
$$d' = a\ b(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})\ c(i)\ d(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}})\ e(-1)\ f(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}})\ g(-i)\ h(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}) \ldots$$

$$\begin{aligned}
S'(t) &= ae^{2\pi it} + b(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})e^{4\pi it} + cie^{6\pi it} + d(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}})e^{8\pi it} + \cdots \\
S'(t-\frac{1}{8}) &= ae^{2\pi i(t-1/8)} + b(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})e^{4\pi i(t-1/8)} + cie^{6\pi i(t-1/8)} + d(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}})e^{8\pi i(t-1/8)} + \cdots \\
&= ae^{2\pi it}e^{-2\pi/8} + b(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})e^{4\pi it}e^{-\pi i/4} + cie^{6\pi it}e^{-3\pi i/4} + d(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}})e^{8\pi it}e^{-\pi i} + \cdots \\
&= a(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})e^{2\pi it} + b(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})e^{4\pi it} + c(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})e^{6\pi it} + d(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})e^{8\pi it} + \cdots \\
S'(t-\frac{1}{8}) &= S(t)(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}) \\
P'(t-\frac{1}{8}) &= P(t)
\end{aligned}$$

□

The first graph below is the power graph of the codeword:
0 2 2 0 1 5 3 3 3 3 5 1 0 2 2 0
The second graph shows the phase shift of the first graph after adding $4x_2+2x_3+x_4$. Note that the second graph shifts back 1/8.





Several other patterns were observed regarding the coefficients of the offsets that reach a max of 3N. There are all eight possible coefficients for x_m, all odd or all even coefficients for x_{m-1}, and two different coefficients for x_{m-3} through x_1. See table. For length 16, there is always a repeated coefficient value in the offset that has a max at $t = 0$. This seems to be related to the occurrence of the first order terms in the coset leader.
For example:
In the coset leader $4x_ax_b + 4x_bx_c + 4x_cx_d - 2x_bx_c + 2x_ax_c$, x_a occurs 6 times, x_b 6 times, x_c 8 times, and x_d 4 times. The 4 offsets that max at $t = 0$ are as follows:
$1x_a + 1x_b + 0x_c + 2x_d$
$1x_a + 1x_b + 4x_c + 6x_d$
$5x_a + 5x_b + 0x_c + 2x_d$
$5x_a + 5x_b + 4x_c + 6x_d$
The coefficients for x_a and x_b are the same and note, x_a and x_b occur the same number of times in the coset leader.

2.2 Length 32 Predictions and Observations

We made several predictions about the length 32 cosets before actually studying them.

1. The same general form for the 3N-coset leader would be used as the form for length 8 and length 16. We were not sure, however, if only the second internal

term would be added or subtracted as was the case with length 8 and 16 or if the middle internal term could be varied.

2. There would be 64 elements in each coset that reached a max of 96.

3. The patterns concerning the offset coefficients would hold.

4. The elements would hit these 96 max's at $t = n/8$.

We tested several combinations of adding and subtracting the internal terms and found that when we added two of the internal terms with certain combinations of other 2nd order RM codes, using the form, $4x_ax_b + 4x_bx_c + 4x_cx_d + 4xdx_e + 2x_bx_c + 2x_ax_c + 2x_cx_d + 2x_cx_e$ the max over the coset was 144 (Not even a multiple of 32). For an example of the 144 coset, see Appendix E. We tried combinations of a single internal term that also had a max greater than 96 over the coset. These failed attempts include adding $2x_bx_c + 2x_cx_e$, $2x_bx_c + 2x_bx_d$, and $2x_bx_c + 2x_cx_d$ to the standard Golay leader. The form, $4x_ax_b + 4x_bx_c + 4x_cx_d + 4xdx_e \pm 2x_bx_c + \pm 2x_ax_c$, always worked.

Our other predictions turned out wrong. The number of elements in the coset that reached a max of 96 varied. We found examples of cosets with 384 and 448 elements that maxed at 96. Since there were more than 64 elements in the coset that maxed at 3N, the previously observed pattern among offset coefficients did not extend to length 32. We also found that most cosets had only elements with a max at $t = n/8$, but there were some cosets with elements with a max at $t = n/16$. Also unlike length 8 and 16, for length 32 some elements had a max of 3N at more than one t value.

Upon examination of the offsets with max at $t = 0$, we found that they matched those of length 8. This led us to believe that there was an inherent difference between codewords of lengths of odd powers of 2 and those of even powers of 2. The proofs in the next section will deal with m-even and m-odd separately.

3 Proofs

This argument will prove the existence that at least one codeword of every coset defined by the general form will have a power maxima of at least $3 * 2^m$. The proofs will use the concatenation lemmas introduced above and the weight theorems dealing with Golay pairs. The power of these codewords will be evaluated at $t = 0$, because this greatly simplifies the calculations. There will be six separate theorems, three pertaining to both the m-even and m-odd case.

3.1 The m-odd Cases

Theorem 6 *The coset of $RM_8(1, m)$ with representative $4x_ax_b + 4x_bx_c + \cdots + 4x_{m-1}x_m + 2x_bx_c + 2x_ax_c$, contains the codeword $4x_ax_b + 4x_bx_c + \cdots + 4x_{m-1}x_m + 2x_bx_c + 2x_ax_c + x_a + 3x_b$ that reaches a power of $3 * 2^m$.*

<u>**Proof:**</u>
At $t = 0$, the complex signal formula simplifies as shown below.

$$
\begin{aligned}
S(t) &= \sum_{n=1}^{N} d_n \cos(2\pi nt) + i \sum_{n=1}^{N} d_n \sin(2\pi nt) \\
S(0) &= \sum_{n=1}^{N} d_n \cos(0) + i \sum_{n=1}^{N} d_n \sin(0) \\
S(0) &= \sum_{n=1}^{N} d_n + i \sum_{n=1}^{N} (0) \\
S(0) &= \sum_{n=1}^{N} d_n
\end{aligned}
$$

At this point d_n is replaced by the $(e^{2\pi i/8})^j$ notation and multiplied by a_j, which is the number of occurrences of js.

$$
\begin{aligned}
S(0) &= \sum_{j=o}^{7} (e^{2\pi i/8})^j a_j \\
S(0) &= \sum_{j=o}^{3} (e^{2\pi i/8})^j a_j + (e^{2\pi i/8})^{j+4} a_{j+4} \\
S(0) &= \sum_{j=o}^{3} (e^{2\pi i/8})^j (a_j + (e^{2\pi i/8})^4 a_{j+4})
\end{aligned}
$$

$$S(0) \quad = \quad \sum_{j=o}^{3}(e^{2\pi i/8})^j(a_j - a_{j+4})$$

To use this formula, the concatenation and weight theorems will be used to calculate the occurrences of each value $(0-7)$ in the codeword.

$4x_1x_2 + 4x_2x_3 + \cdots + 4x_{m-1}x_m + 2x_2x_3 + 2x_1x_3 + x_1 + 3x_2$
$= [L_{m-1} + 2x_1x_2 + 3x_1 \mid L_{m-1} + 7x_1 + 2x_1x_2 + 2x_2 + \mathbf{1}]$
$= [L_{m-2} \mid L_{m-2} + 6x_1 + 3(\mathbf{1}) \mid L_{m-2} + 2x_1 + \mathbf{1} \mid L_{m-2}]$
$= [\underbrace{L_{m-2}}_{a} \mid \underbrace{L_{m-3} + 3(\mathbf{1})}_{b} \mid \underbrace{L_{m-3} + 4x_1 + \mathbf{1}}_{c} \mid \underbrace{L_{m-3} + \mathbf{1}}_{d} \mid \underbrace{L_{m-3} + 4x_1 + 3(\mathbf{1})}_{e} \mid \underbrace{L_{m-2}}_{f}]$

The codeword is now broken down into eighths. Observe that the first and last quarters, a and f, of the codeword are the same. From Theorem 2, the weight for a coset leader is $2^{m-1} - 2^{\frac{m-1}{2}}$. At $m-2$ this will give $2^{m-3} - 2^{\frac{m-3}{2}}$. Since there are two of these (the first and last quarters) the weight will be:

$$2 * [2^{m-3} - 2^{\frac{m-3}{2}}] = 2^{m-2} - 2^{\frac{m-1}{2}}$$

The coset leader is multiplied by 4, so the weight gives the number of 4s instead of the number of 1s. The number of 0s is calculated by subtracting the number of 4s from the length and multiplying by two because there are two quarters containing 0s and 4s.

$$2[2^{m-2} - (2^{m-3} - 2^{\frac{m-3}{2}})] = 2^{m-1} - 2^{m-2} + 2^{\frac{m-1}{2}}$$

When computing the power of a codeword at $t = 0$, 0s and 4s cancel out because their values are 1 and -1 respectively, as demonstrated in the simplified signal formula. There will be a greater amount of 0s so all the 4s will be cancelled and only 0s will remain. The amount of 0s remaining after cancellation is found by subtracting the number of 4s from the number of 0s.

$$2^{m-1} - 2^{m-2} + 2^{\frac{m-1}{2}} - [2^{m-2} - 2^{\frac{m-1}{2}}] = 2(2^{\frac{m-1}{2}}) = 2^{\frac{m+1}{2}}$$

Given the number of 0s, 1s, and 3s the power is easily calculated at $t = 0$.

$$S(0) \quad = \quad 2^{\frac{m+1}{2}} * (e^{2\pi i/8})^0) + 2^{\frac{m-1}{2}} * (e^{2\pi i/8})^1) + 2^{\frac{m-1}{2}} * (e^{2\pi i/8})^3)$$
$$S(0) \quad = \quad 2^{\frac{m-1}{2}}(2(e^{2\pi i/8})^0 + (e^{2\pi i/8})^1 + (e^{2\pi i/8})^3$$

The previous argument concerning permutations applies to this proof. Therefore any permutation of the general form $4x_ax_b + 4x_bx_c + \cdots + 4x_{m-1}x_m + 2x_bx_c + 2x_ax_c + x_a + 3x_b$ will also be a coset that contains at least one codeword with a power maxima at $3 * 2^m$.

The other two general forms are proved in the same way using concatenation lemmas and Golay pair weight theorems. Since they are so similar to the preceding proof, the theorems will simply be stated below.

Theorem 7 *The coset of* $RM_8(1,m)$ *with representative* $4x_ax_b + 4x_bx_c + \cdots + 4x_{m-1}x_m + 2x_bx_c - 2x_ax_c$ *contains codeword,* $4x_ax_b + 4x_bx_c + \cdots + 4x_{m-1}x_m + 2x_bx_c - 2x_ax_c + 3x_a + 3x_b + 2x_c$ *that reaches a power of* $3 * 2^m$*.*

This codeword is comprised of 3's, 2's and 0's rather than with 0's, 1's and 3's as in the previous proof. This changes the signal formula to the following.

$$S(0) = 2^{\frac{m+1}{2}} * (e^{2\pi i/8})^3) + 2^{\frac{m-1}{2}} * (e^{2\pi i/8})^2) + 2^{\frac{m-1}{2}} * (e^{2\pi i/8})^0)$$

Theorem 8 *The coset of* $RM_8(1,m)$ *with representative* $4x_ax_b + 4x_bx_c + \cdots + 4x_{m-1}x_m - 2x_bx_c + 2x_ax_c$ *contains the codeword,* $4x_ax_b + 4x_bx_c + \cdots + 4x_{m-1}x_m - 2x_bx_c + 2x_ax_c + x_a + x_b + 6x_c$ *that reaches a power of* $3 * 2^m$*.*

This codeword is comprised of 1's, 0's, and 6's, resulting in the below signal formula.

$$S(0) = 2^{\frac{m+1}{2}} * (e^{2\pi i/8})^1) + 2^{\frac{m-1}{2}} * (e^{2\pi i/8})^0) + 2^{\frac{m-1}{2}} * (e^{2\pi i/8})^6)$$

3.2 The m-even Cases

For the three general forms in the m-even case, the proofs work in very much the same way. They follow the same pattern of decomposing the codewords by using the concatenation lemmas and then using the weight theorems to count the values in the codeword. The only difference with the m-even proofs is that the codewords must be broken down one more step than the m-odd codewords. This results in a slight variation in the weight formulas. Many of the values cancel with their compliments just as in the preceding proof. Then there are always four positions of three different values. Unlike the m-odd cases, these values all occur the same number of times. For example, there will be four 0s, four 1s, and four 3s left. Using the complex signal equation and then the power equation will always yield a power of $3 * 2^m$. Since the proof procedure is practically the same, the theorems for each general case will be listed below, along with the offsets needed to reach a power of $3*2^m$ at $t = 0$.

Theorem 9 *The coset of* $RM_8(1,m)$ *with representative* $4x_ax_b + 4x_bx_c + \cdots + 4x_{m-1}x_m + 2x_bx_c + 2x_ax_c$ *contains the codeword,* $4x_ax_b + 4x_bx_c + \cdots + 4x_{m-1}x_m + 2x_bx_c + 2x_ax_c + x_a + 3x_b + 2x_c + 2x_d$ *that reaches a power of* $3 * 2^m$*.*

This codeword contains 0's, 2's, and 3's, resulting in the below signal formula.

$$S(0) = 2^{\frac{m}{2}} * (e^{2\pi i/8})^0) + 2^{\frac{m}{2}} * (e^{2\pi i/8})^2) + 2^{\frac{m}{2}} * (e^{2\pi i/8})^3)$$

Theorem 10 *The coset of $RM_8(1,m)$ with representative $4x_ax_b + 4x_bx_c + \cdots + 4x_{m-1}x_m + 2x_bx_c - 2x_ax_c$ contains the codeword, $4x_ax_b + 4x_bx_c + \cdots + 4x_{m-1}x_m + 2x_bx_c - 2x_ax_c + 3x_a + 3x_b + 6x_d$, that reaches a power of $3 * 2^m$.*

This codeword contains 0's, 1's, and 3's, resulting in the below signal formula.

$$S(0) = 2^{\frac{m}{2}} * (e^{2\pi i/8})^0) + 2^{\frac{m}{2}} * (e^{2\pi i/8})^1) + 2^{\frac{m}{2}} * (e^{2\pi i/8})^3)$$

Theorem 11 *The coset of $RM_8(1,m)$ with representative $4x_ax_b + 4x_bx_c + \cdots + 4x_{m-1}x_m - 2x_bx_c + 2x_ax_c$ contains the codeword, $4x_ax_b + 4x_bx_c + \cdots + 4x_{m-1}x_m - 2x_bx_c + 2x_ax_c + x_a + x_b + 2x_d$ that reaches a power of $3 * 2^m$.*

This codeword contains the same numbers as the previous codeword, but in a different order. This does not change the signal equation.

4 Further Investigation

The preceding proof and theorems prove the existence of one codeword that reaches a power maxima of at least $3 * 2^m$ in each of the general form cosets for both the m-odd and m-even cases. Adding $\beta(4x_{m-2} + 2x_{m-1} + x_m)$, where $\beta = 0, 1, 2, \ldots 7$, to each of the general forms and their offsets for maxima at $t = 0$ yields 8 more codewords in each coset that reach the $3 * 2^m$ maxima. It would be useful to have an algorithm to generate all the codewords in a coset that reached the given integer maxima. Although these theorems prove the existence of a codeword that reaches the integer maxima, they do not prove that this maxima is an upper bound over the coset. A proof to this affect is necessary in completing the study of codewords of good power properties.

The general forms derived from the Golay pair formulas are very useful in characterizing the cosets that have a power maxima of $3 * 2^m$. There doesn't seem to be any other combination of elements besides $\pm 2x_b x_c \pm 2x_a x_c$ that yields the unique power maxima of $3 * 2^m$. We tried different combinations in length 32. It would be worth investigating further if these are in fact the only combinations possible. It seems probable that some sort of proof could be constructed to argue this property.

During the investigation of codewords of length 2^5, an additional integer power maxima of 144 was observed. Since 144 is not a power of 2, it is improbable that these power maximas were a result of any Golay pairs or Golay quadruples. It can only be assumed that there may be another intermediate group of cosets that have similar properties to those discussed in this paper. The cosets introduced in this paper along with the possibility of other existing groups of cosets that reach integer power maxima should be further researched. Increased understanding of these unique cosets and their codewords would be extremely valuable.

Appendix A
List of Coset Leaders for Length 16

Below is a list of all the coset leaders in length 16 over Z_8 that have a max power of 48 over the coset. There are exactly 48 coset leaders that have this property. This agrees with the argument that there are $2 * m!$ coset leaders for length 2^m. For further explanation, refer to Section 2.1.1. Note that the coset leaders are doubled in calculation (refer to Section 1.2.2.).

x_1x_2	x_1x_3	x_1x_4	x_2x_3	x_2x_4	x_3x_4
1	0	2	2	1	0
1	2	1	0	2	0
0	0	2	2	1	1
0	1	2	2	0	1
1	2	0	1	2	0
0	2	0	1	2	1
0	2	1	0	2	1
2	0	0	1	1	2
2	0	1	0	1	2
2	1	0	1	0	2
1	1	2	2	0	0
2	1	1	0	0	2
0	2	3	0	2	3
2	0	0	3	3	2
2	0	3	0	3	2
2	3	0	3	0	2
2	3	3	0	0	2
3	2	0	3	2	0
3	2	3	0	2	0
3	3	2	2	0	0
0	0	2	2	1	3
0	0	2	2	3	1
0	0	2	2	3	3
0	2	0	3	2	3
0	2	1	0	2	3
0	3	2	2	0	3
1	3	2	2	0	0
3	0	2	2	3	0
0	2	0	3	2	1
0	1	2	2	0	3
2	0	0	1	3	2
2	0	1	0	3	2

2	1	0	3	0	2
1	0	2	2	3	0
1	2	3	0	2	0
0	2	0	1	2	3
0	2	3	0	2	1
0	3	2	2	0	1
2	0	0	3	1	2
2	0	3	0	1	2
2	1	3	0	0	2
2	3	1	0	0	2
1	2	0	3	2	0
3	0	2	2	1	0
3	1	2	2	0	0
2	3	0	1	0	2
3	2	0	1	2	0
3	2	1	0	2	0
0	0	2	2	0	1
0	2	0	0	2	1

Appendix B
Within a Coset of Length 8

This table shows all the offsets that when added to the coset leader

$$2x_1x_2 + 4x_1x_3 + 2x_2x_3$$

give a power of 24. This is an example of length 8 codewords over Z_8 that reach a power of $3 * 2^m$. Note that each multiple of 1/8 occurs twice, which means there are two different sets of 8 phase shifts. Therefore there are 16 elements in the coset that reach a power of 24.

x1	x2	x3	
1	0	2	Max: 24 at: 0.375
1	0	6	Max: 24 at: 0.875
1	2	3	Max: 24 at: 0.75
1	2	7	Max: 24 at: 0.25
1	4	0	Max: 24 at: 0.625
1	4	4	Max: 24 at: 0.125
1	6	1	Max: 24 at: 0
1	6	5	Max: 24 at: 0.5
5	0	2	Max: 24 at: 0.875
5	0	6	Max: 24 at: 0.375
5	2	3	Max: 24 at: 0.25
5	2	7	Max: 24 at: 0.75
5	4	0	Max: 24 at: 0.125
5	4	4	Max: 24 at: 0.625
5	6	1	Max: 24 at: 0.5
5	6	5	Max: 24 at: 0

Appendix C
Within a Coset of Length 16

This table shows all the offsets that when added to the coset leader

$$2x_1x_2 + 4x_1x_4 + 4x_2x_3 + 2x_2x_4$$

give a power of 48. This is an example of length 16 codewords over Z_8 that reach a power of $3 * 2^m$. Note that each multiple of 1/8 occurs four times, which means there are four different sets of 8 phase shifts. Therefore there are 32 elements in the coset that reach a power of 48.

x1	x2	x3	x4	
1	0	0	2	Max: 48 at: 0.875
1	0	0	6	Max: 48 at: 0.375
1	0	2	1	Max: 48 at: 0
1	0	2	5	Max: 48 at: 0.5
1	0	4	0	Max: 48 at: 0.125
1	0	4	4	Max: 48 at: 0.625
1	0	6	3	Max: 48 at: 0.75
1	0	6	7	Max: 48 at: 0.25
1	4	0	0	Max: 48 at: 0.125
1	4	0	4	Max: 48 at: 0.625
1	4	2	3	Max: 48 at: 0.75
1	4	2	7	Max: 48 at: 0.25
1	4	4	2	Max: 48 at: 0.875
1	4	4	6	Max: 48 at: 0.375
1	4	6	1	Max: 48 at: 0
1	4	6	5	Max: 48 at: 0.5
5	0	0	2	Max: 48 at: 0.375
5	0	0	6	Max: 48 at: 0.875
5	0	2	1	Max: 48 at: 0.5
5	0	2	5	Max: 48 at: 0
5	0	4	0	Max: 48 at: 0.625
5	0	4	4	Max: 48 at: 0.125
5	0	6	3	Max: 48 at: 0.25
5	0	6	7	Max: 48 at: 0.75
5	4	0	0	Max: 48 at: 0.625
5	4	0	4	Max: 48 at: 0.125
5	4	2	3	Max: 48 at: 0.25
5	4	2	7	Max: 48 at: 0.75

5	4	4	2	Max: 48 at: 0.375
5	4	4	6	Max: 48 at: 0.875
5	4	6	1	Max: 48 at: 0.5
5	4	6	5	Max: 48 at: 0

The following table is another example of a coset of length 16 over Z_8. The offsets shown are those that when added to the coset leader

$$2x_1x_2 + 4x_1x_3 + 2x_1x_4 + 4x_2x_4$$

give a power of 48. As in the preceding table, there are 32 elements in the coset that reach a power of $3 * 2^m$ and four different sets of phase shifts.

x1	x2	x3	x4	
0	1	0	0	Max: 48 at: 0.625
0	1	0	4	Max: 48 at: 0.125
0	1	2	1	Max: 48 at: 0
0	1	2	5	Max: 48 at: 0.5
0	1	4	2	Max: 48 at: 0.375
0	1	4	6	Max: 48 at: 0.875
0	1	6	3	Max: 48 at: 0.75
0	1	6	7	Max: 48 at: 0.25
0	5	0	0	Max: 48 at: 0.125
0	5	0	4	Max: 48 at: 0.625
0	5	2	1	Max: 48 at: 0.5
0	5	2	5	Max: 48 at: 0
0	5	4	2	Max: 48 at: 0.875
0	5	4	6	Max: 48 at: 0.375
0	5	6	3	Max: 48 at: 0.25
0	5	6	7	Max: 48 at: 0.75
4	1	0	2	Max: 48 at: 0.375
4	1	0	6	Max: 48 at: 0.875
4	1	2	3	Max: 48 at: 0.75
4	1	2	7	Max: 48 at: 0.25
4	1	4	0	Max: 48 at: 0.625
4	1	4	4	Max: 48 at: 0.125
4	1	6	1	Max: 48 at: 0
4	1	6	5	Max: 48 at: 0.5
4	5	0	2	Max: 48 at: 0.875
4	5	0	6	Max: 48 at: 0.375
4	5	2	3	Max: 48 at: 0.25
4	5	2	7	Max: 48 at: 0.75
4	5	4	0	Max: 48 at: 0.125
4	5	4	4	Max: 48 at: 0.625
4	5	6	1	Max: 48 at: 0.5
4	5	6	5	Max: 48 at: 0

Appendix D
Within a Coset of Length 32

This abbreviated table shows some of the offsets that when added to the coset leader

$$4x_1x_2 + 2x_2x_4 + 4x_4x_5 + 4x_3x_5 - 2x_1x_4$$

give a power of 96. This is an example of length 32 codewords over Z_8 that reach a power of $3{*}2^m$. Note that in addition to the multiples of 1/8, there are also multiples of 1/16 at which the power reaches 96. After investigating several examples of cosets of length 32, it was evident that there was not a set number of elements in a coset; it was either 384 or 448. The elements that have a power max of 96 at multilples of 1/16 may be the cause of this irregularity.

#	x1	x2	x3	x4	x5	
1	3	1	0	0	0	Max: 96 at: 0
2	3	1	0	0	1	Max: 96 at: 0
3	3	1	0	0	2	Max: 96 at: 0
4	3	1	0	0	3	Max: 96 at: 0
5	3	1	0	0	4	Max: 96 at: 0
6	3	1	0	0	5	Max: 96 at: 0.5
7	3	1	0	0	6	Max: 96 at: 0
8	3	1	0	0	7	Max: 96 at: 0.5
9	3	1	0	2	0	Max: 96 at: 0.125
10	3	1	0	2	1	Max: 96 at: 0.625
11	3	1	0	2	2	Max: 96 at: 0.125
.						
.						
.						
36	3	1	1	1	7	Max: 96 at: 0.125
37	3	1	1	3	2	Max: 96 at: 0.75
38	3	1	1	3	3	Max: 96 at: 0.125
39	3	1	1	3	6	Max: 96 at: 0.25
40	3	1	1	3	7	Max: 96 at: 0.625
41	3	1	1	5	1	Max: 96 at: 0.875
42	3	1	1	5	2	Max: 96 at: 0.25
43	3	1	1	5	5	Max: 96 at: 0.375
.						
.						
.						
161	3	5	2	1	0	Max: 96 at: 0.4375

162	3	5	2	1	1	Max: 96 at: 0.9375
163	3	5	2	1	2	Max: 96 at: 0.4375
164	3	5	2	1	3	Max: 96 at: 0.4375
165	3	5	2	1	4	Max: 96 at: 0.9375
166	3	5	2	1	5	Max: 96 at: 0.4375
167	3	5	2	1	6	Max: 96 at: 0.9375
168	3	5	2	1	7	Max: 96 at: 0.4375
169	3	5	2	3	0	Max: 96 at: 0.5625
170	3	5	2	3	1	Max: 96 at: 0.0625
171	3	5	2	3	2	Max: 96 at: 0.0625
172	3	5	2	3	3	Max: 96 at: 0.5625
173	3	5	2	3	4	Max: 96 at: 0.0625
.						
.						
.						
448	7	5	7	7	5	Max: 96 at: 0.375

Appendix E
Coset with Max of 144

This table shows all the offsets that when added to the coset leader

$$4x_1x_3 + 6x_1x_5 + 6x_4x_5 + 4x_2x_4 + 2x_2x_5 + 2x_3x_5$$

give a power of 144. This is an example of length 32 codewords over Z_8.

x1	x2	x3	x4	x5	
0	0	0	2	3	Max: 144 at: 0.625
0	0	0	2	7	Max: 144 at: 0.125
0	0	0	6	1	Max: 144 at: 0.875
0	0	0	6	5	Max: 144 at: 0.375
0	0	4	0	2	Max: 144 at: 0.75
0	0	4	0	6	Max: 144 at: 0.25
0	0	4	4	0	Max: 144 at: 0
0	0	4	4	4	Max: 144 at: 0.5
0	4	0	2	3	Max: 144 at: 0.875
0	4	0	2	7	Max: 144 at: 0.375
0	4	0	6	1	Max: 144 at: 0.125
0	4	0	6	5	Max: 144 at: 0.625
0	4	4	0	2	Max: 144 at: 0
0	4	4	0	6	Max: 144 at: 0.5
0	4	4	4	0	Max: 144 at: 0.25
0	4	4	4	4	Max: 144 at: 0.75
1	7	3	1	2	Max: 144 at: 0.75
1	7	3	1	6	Max: 144 at: 0.25
1	7	3	5	0	Max: 144 at: 0
1	7	3	5	4	Max: 144 at: 0.5
1	7	7	3	3	Max: 144 at: 0.625
1	7	7	3	7	Max: 144 at: 0.125
1	7	7	7	1	Max: 144 at: 0.875
1	7	7	7	5	Max: 144 at: 0.375
2	2	2	2	2	Max: 144 at: 0
2	2	2	2	6	Max: 144 at: 0.5
2	2	2	6	0	Max: 144 at: 0.25
2	2	2	6	4	Max: 144 at: 0.75
2	2	6	0	1	Max: 144 at: 0.125
2	2	6	0	5	Max: 144 at: 0.625
2	2	6	4	3	Max: 144 at: 0.875
2	2	6	4	7	Max: 144 at: 0.375

2	6	2	2	2	Max: 144 at: 0.75
2	6	2	2	6	Max: 144 at: 0.25
2	6	2	6	0	Max: 144 at: 0
2	6	2	6	4	Max: 144 at: 0.5
2	6	6	0	1	Max: 144 at: 0.875
2	6	6	0	5	Max: 144 at: 0.375
2	6	6	4	3	Max: 144 at: 0.625
2	6	6	4	7	Max: 144 at: 0.125
3	1	1	3	0	Max: 144 at: 0
3	1	1	3	4	Max: 144 at: 0.5
3	1	1	7	2	Max: 144 at: 0.75
3	1	1	7	6	Max: 144 at: 0.25
3	1	5	1	3	Max: 144 at: 0.625
3	1	5	1	7	Max: 144 at: 0.125
3	1	5	5	1	Max: 144 at: 0.875
3	1	5	5	5	Max: 144 at: 0.375
3	5	1	3	2	Max: 144 at: 0.75
3	5	1	3	6	Max: 144 at: 0.25
3	5	1	7	0	Max: 144 at: 1
3	5	1	7	4	Max: 144 at: 0.5
3	5	5	1	1	Max: 144 at: 0.875
3	5	5	1	5	Max: 144 at: 0.375
3	5	5	5	3	Max: 144 at: 0.625
3	5	5	5	7	Max: 144 at: 0.125
4	0	0	0	0	Max: 144 at: 0.75
4	0	0	0	4	Max: 144 at: 0.25
4	0	0	4	2	Max: 144 at: 0.5
4	0	0	4	6	Max: 144 at: 0
4	0	4	2	1	Max: 144 at: 0.625
4	0	4	2	5	Max: 144 at: 0.125
4	0	4	6	3	Max: 144 at: 0.375
4	0	4	6	7	Max: 144 at: 0.875
4	4	0	0	0	Max: 144 at: 0
4	4	0	0	4	Max: 144 at: 0.5
4	4	0	4	2	Max: 144 at: 0.75
4	4	0	4	6	Max: 144 at: 0.25
4	4	4	2	1	Max: 144 at: 0.875
4	4	4	2	5	Max: 144 at: 0.375
4	4	4	6	3	Max: 144 at: 0.625
4	4	4	6	7	Max: 144 at: 0.125
5	3	3	3	1	Max: 144 at: 0.875

5	3	3	3	5	Max: 144 at: 0.375
5	3	3	7	3	Max: 144 at: 0.625
5	3	3	7	7	Max: 144 at: 0.125
5	3	7	1	0	Max: 144 at: 0
5	3	7	1	4	Max: 144 at: 0.5
5	3	7	5	2	Max: 144 at: 0.75
5	3	7	5	6	Max: 144 at: 0.25
6	2	2	0	3	Max: 144 at: 0.625
6	2	2	0	7	Max: 144 at: 0.125
6	2	2	4	1	Max: 144 at: 0.875
6	2	2	4	5	Max: 144 at: 0.375
6	2	6	2	0	Max: 144 at: 0
6	2	6	2	4	Max: 144 at: 0.5
6	2	6	6	2	Max: 144 at: 0.75
6	2	6	6	6	Max: 144 at: 0.25
6	6	2	0	3	Max: 144 at: 0.375
6	6	2	0	7	Max: 144 at: 0.875
6	6	2	4	1	Max: 144 at: 0.625
6	6	2	4	5	Max: 144 at: 0.125
6	6	6	2	0	Max: 144 at: 0.75
6	6	6	2	4	Max: 144 at: 0.25
6	6	6	6	2	Max: 144 at: 0.5
6	6	6	6	6	Max: 144 at: 0
7	1	1	1	3	Max: 144 at: 0.625
7	1	1	1	7	Max: 144 at: 0.125
7	1	1	5	1	Max: 144 at: 0.875
7	1	1	5	5	Max: 144 at: 0.375
7	1	5	3	0	Max: 144 at: 0
7	1	5	3	4	Max: 144 at: 0.5
7	1	5	7	2	Max: 144 at: 0.75
7	1	5	7	6	Max: 144 at: 0.25
7	5	1	1	1	Max: 144 at: 0.875
7	5	1	1	5	Max: 144 at: 0.375
7	5	1	5	3	Max: 144 at: 0.625
7	5	1	5	7	Max: 144 at: 0.125
7	5	5	3	2	Max: 144 at: 0.75
7	5	5	3	6	Max: 144 at: 0.25
7	5	5	7	0	Max: 144 at: 0
7	5	5	7	4	Max: 144 at: 0.5

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