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Centers and Shore Points in λ-Dendroids

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IN λ-DENDROIDS

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ABSTRACT. A dendroid is the disjoint union of the set of centers and the set of shore points. We show this is also true for λ-dendroids and use this fact to show that the finite union of shore continua in a λ-dendroid is a shore set.

1. INTRODUCTION

The purpose of this paper is to show that a λ-dendroid is the disjoint union of the set of shore points and the set of centers and to use this to generalize to λ-dendroids various results about shore sets in dendroids. A continuum is a compact, connected, metric space. A dendroid is a hereditarily unicoherent, arc connected continuum. It follows that a dendroid is hereditarily decomposable. A λ-dendroid is a hereditarily unicoherent, hereditarily decomposable continuum. The closure of the graph of $f(x) = sin(1/x)$ for $0 < x \leq 1$ is an example of a λ-dendroid that is not a dendroid. A point x of a continuum X is a shore point, as defined in [6], if $X \setminus \{x\}$ contains continua arbitrarily close to X in the Hausdorff metric, and a set A is a shore set if $X \setminus A$ contains continua arbitrarily close to X in the Hausdorff metric. Alternately, we could say a point x of a continuum X is a shore point if for each finite collection $\mathcal{C}$ of open sets in X there is a continuum in $X \setminus \{x\}$ that intersects each element of $\mathcal{C}$. As defined in [2], a point x is a strong

center of a continuum X if there are open sets U and V in X such that every continuum that intersects U and V contains x. It is clear then that no point can be both a strong center and a shore point.

As defined in [5], a point x is a center in the continuum X if there are two points u and v in X, called basin points for x, such that for each $\epsilon > 0$, there is a continuum B containing x, called a bottleneck for x, such that $diam(B) < \epsilon$, and there are two open sets U and V with $u \in U$ and $v \in V$ such that every continuum that contains a point in U and a point in V also contains a point in B. It is known that every dendroid has at least one center point [5, Theorem 3.6]. This is not true for λ-dendroids as seen in the example below. Note that a strong center x is just a center with a bottleneck equal to $\{x\}$ for each ϵ.

Piotr Minc suggested this example of a λ-dendroid with no center. Start with Example 2 in [4], which is a dendroid X with one center c. Then the λ-dendroid Y in Example 4 of [4] is the dendroid X with the center c removed and compactified by adding an arc L so that Y has continuum many arc components each of which is dense in Y. Let x be any point in Y, and let $\epsilon > 0$ such that the ball of radius ϵ about x does not contain the arc L. If B is a continuum that contains x and has diameter less than ϵ, then B is arc connected. Since Y has infinitely many dense arc components, if U and V are open sets in Y, there is an arc component of Y that does not contain B but does contain an element of U and an element of V. Therefore, B cannot be a bottleneck. It follows that no point of Y is a center.

Recent work in this area by the author has produced a fairly complete picture of structure of dendroids with respect to shore points and centers [7]. For a continuum X, let $S(X)$ be the set of shore points in X and let $SC(X)$ be the set of strong centers of X. Let $Cn(X)$ be the set of all centers in X. It is shown in [7] that for a dendroid X if $SC(X) \neq \emptyset$, then $X \setminus S(X) = SC(X)$. It is also shown that if $SC(X) = \emptyset$, then X is a very special kind of dendroid with a rich structure.

J. J. Charatonik has shown that for each λ-dendroid X there is a unique minimal monotone map g of X onto a dendroid [1]. The map g is minimal in the sense that if h is another monotone map of X onto a dendroid, then $g^{-1}(g(x)) \subset h^{-1}(h(x))$ for each $x \in X$. Charatonik calls $g^{-1}(g(x))$ the stratum of x. For the closure of the

graph of $f(x) = sin(1/x)$ for $0 < x \leq 1$, the points with $x = 0$ are all in the same stratum, and every other point is in a stratum with only one point. Note that a λ-dendroid X is a dendroid if and only if every stratum has only one point. It will be shown in this paper that, for a λ-dendroid X, if $SC(X)$ intersects more than one stratum of X, then $X \setminus S(X) = SC(X)$.

For a subset A of a λ-dendroid X, let A^* be the union of all strata of X that intersect A. If x and y are points in a λ-dendroid X, then $I(x, y)$ will refer to the unique continuum in X irreducible with respect to containing x and y. A subset A of X is λ-connected if $I(x, y) \subset A$ for each x and y in A. A λ-component of a set A in X is a maximal λ-connected subset of A. We will show that the union of a finite collection of λ-components of $X \setminus SC(X)^*$ is a shore set. It follows that, for a dendroid D, the union of a finite collection of arc components of $D \setminus SC(D)$ is a shore set, a result first proven in [7, Theorem 4]. Also, Alejandro Illanes has shown that the union of a finite pairwise disjoint collection of shore continua in a dendroid is a shore set [3, Theorem 3], and we will prove that in a λ-dendroid the union of a finite collection of shore continua, $\{K_i\}$, such that $K_i^* \cap K_j^* \cap SC(X) = \emptyset$ for each $i \neq j$ is a shore set. Again the result of Illanes for dendroids follows from this theorem.

2. Shore points and strong centers

The first two lemmas follow immediately from the definitions.

Lemma 2.1. *If X is a λ-dendroid, then $S(X) \cap SC(X) = \emptyset$.*

Lemma 2.2. *If X is a λ-dendroid, then $X \setminus S(X)$ is λ-connected.*

An important fact about the structure of λ-dendroids that relates to centers and which will be used extensively in later sections of this paper is that $SC(X)^*$ is λ-connected. From the two previous lemmas it is seen that one way to establish this fact is by showing that $X \setminus S(X) = SC(X)$ or $SC(X)$ is contained in a single stratum of X, which is done below in Corollary 2.9.

The following simple lemma will be used many times. Note that if g is the minimal monotone map then it says that $I(x, y) = I(x, z)$ implies y and z are in the same stratum, that is $y^* = z^*$.

Lemma 2.3. *If X is a λ-dendroid, $g : X \to D$ is a monotone map from X onto the dendroid D, and $x, y, z \in X$ such that $I(x, y) = I(x, z)$, then $g(y) = g(z)$.*

Proof: Suppose X is a λ-dendroid, $g : X \to D$ is a monotone map from X onto the dendroid D, and $x, y, z \in X$ such that $I(x, y) = I(x, z)$. Let A be the arc in D from $g(x)$ to $g(y)$ and let B be the arc in D from $g(x)$ to $g(z)$. Then $g^{-1}(A)$ is a continuum containing x and y, so $I(x, y) \subset g^{-1}(A)$, and therefore, $g(I(x, y)) \subset A$. But $g(I(x, y))$ is a continuum containing $g(x)$ and $g(y)$ so $A \subset g(I(x, y))$. So $A = g(I(x, y))$, and similarly, $B = g(I(x, z))$. So $A = B$, and therefore, $g(y) = g(z)$. □

If $x \in X$ and $A \subset X$, define $Q_x(A) = \{z \in X | I(x, z) \cap A \neq \emptyset\}$. We see in the next few lemmas that the interior of $Q_x(y)$ plays an important role in determining whether or not y is a shore point. It is easy to see that $X \setminus Q_x(y)$ is λ-connected. So if $X \setminus Q_x(y)$ is also dense, then it is easy to see that there is a continuum in $X \setminus \{y\}$ that is close to X, showing that y is a shore point. But the next lemma turns this around in certain cases to show that if $int(Q_x(y)) \neq \emptyset$, then y is a strong center.

Lemma 2.4. *If $g : X \to D$ is a monotone map from the λ-dendroid X onto the dendroid D, $x \in X$ such that either $x \in SC(X)$ or $g(x) \in Cn(D)$, $y \in X$ such that $g(x) \neq g(y)$, and $int(Q_x(y)) \neq \emptyset$, then $y \in SC(X)$.*

Proof: Suppose $g : X \to D$ is a monotone map from the λ-dendroid X onto the dendroid D. Let x and y be elements of X such that $g(x) \neq g(y)$. Suppose $x \in SC(X)$ and $int(Q_x(y)) \neq \emptyset$. Let U and V be basins for x, and let O be an open set in $Q_x(y)$. Let u and v be points in U and V, respectively, and let o_1 and o_2 be points in O. Then $I(u, o_1) \cup I(o_1, y) \cup I(y, o_2) \cup I(o_2, v)$ is a continuum that intersects U and V so it must contain x. If $x \in I(o_1, y)$, then, since $y \in I(o_1, x)$, $I(0_1, y) = I(0_1, x)$. Thus, by Lemma 2.3, $g(x) = g(y)$. But $g(x) \neq g(y)$. So x is not in $I(o_1, y)$, and similarly x is not in $I(y, o_2)$. So either $x \in I(u, o_1)$ or $x \in I(v, o_2)$. But if $x \in I(u, o_1)$, then $y \in I(u, o_1)$, and if $x \in I(v, o_2)$, then $y \in I(v, o_2)$ since o_1 and o_2 are both in $Q_x(y)$. So we have that either $y \in I(u, o_1)$ or $y \in I(v, o_2)$. But the elements u,

v, o_1, and o_2 were chosen arbitrarily, so either U and O are basins for y, or V and O are basins for y. So $y \in SC(X)$.

Next suppose $g(x) \in Cn(D)$ and $int(Q_x(y)) \neq \emptyset$. Let B be a bottleneck for $g(x)$ that does not contain $g(y)$. Let U and V be basins for B. Let O be an open set in $Q_x(y)$, and let u and v be points in $g^{-1}(U)$ and $g^{-1}(V)$, and let o_1 and o_2 be points in O. Then $g(I(u,o_1) \cup I(o_1,y) \cup I(y,o_2) \cup I(o_2,v)) \cap B \neq \emptyset$, so $I(u,0_1) \cup I(o_1,y) \cup I(y,o_2) \cup I(o_2,v) \cap g^{-1}(B) \neq \emptyset$. If there is a w in $g^{-1}(B) \cap I(0_1,y)$, then $I(o_1,w) \subset I(o_1,y)$, and by Lemma 2.3, $g(y) = g(w)$. Similarly, $g(y) = g(w)$ if $w \in g^{-1}(B) \cap I(y,o_2)$. So, since $g(y) \neq g(w)$ for each $w \in g^{-1}(B)$, $g^{-1}(B) \cap (I(o_1,y) \cup I(y,o_2)) = \emptyset$. So either $y \in I(u,o_1)$ or $y \in I(v,o_2)$. So, as in the argument above, either $g^{-1}(U)$ and O are basins for y, or $g^{-1}(V)$ and O are basins for y. Thus, $y \in SC(X)$. $\square$

Lemma 2.5. *If F is a subset of a λ-dendroid X such that $int(F) = \emptyset$ and $X \setminus F$ is λ-connected, then F is a shore set. Also, if A is a shore set and $x \in SC(X)$, then $int(Q_x(A)) = \emptyset$.*

Proof: Assume F is a subset of a λ-dendroid X such that $int(F) = \emptyset$ and $X \setminus F$ is λ-connected. So for every $\epsilon > 0$ there is a continuum K in $X \setminus F$ such that for each $y \in X$ there is a $k \in K$ such that $d(x,k) < \epsilon$.

Now assume A is a shore set in X and $x \in SC(X)$. Let U and V be basins for x . There is a sequence of continua in $X \setminus A$ that converges in the Hausdorff metric to X and if a continuum is sufficiently close to X in the Hausdorff metric, it will intersect both U and V and therefore contain x. It follows that the λ-component of x in $X \setminus A$ is dense in X. So $int(Q_x(A) = \emptyset$. $\square$

For continua A and B in a λ-dendroid X, it is easy to prove that there is a unique continuum in X that is irreducible with respect to containing a point in A and a point in B. The notation $I(A,B)$ will be used to refer to that continuum.

Lemma 2.6. *If X is a λ-dendroid such that $SC(X) \neq \emptyset$ and K is a continuum in $S(X)$, then K is a shore continuum.*

Proof: Assume X is a λ-dendroid such that $SC(X) \neq \emptyset$ and K is a continuum in $S(X)$. Let $x \in SC(X)$ and let $z \in I(x,K) \cap K$. Then $int(Q_x(z)) = \emptyset$ by Lemma 2.5, and so $X \setminus Q_x(z)$ is dense in

X and clearly λ-connected. It follows that $Q_x(z)$ is a shore set, and therefore, K is a shore set since $K \subset Q_x(z)$. ☐

Lemma 2.7. *Suppose $g : X \to D$ is a monotone map from the λ-dendroid X onto the dendroid D and K is a subcontinuum of X such that $K \cap SC(X) = \emptyset$. If there exists $x \in X$ such that $g(x) \in D \setminus g(K)$ and either $x \in SC(X)$ or $g(x) \in Cn(D)$, then K is a shore set.*

Proof: Suppose $g : X \to D$ is a monotone map from the λ-dendroid X onto the dendroid D, $SC(X) \neq \emptyset$, and K is a subcontinuum of $X \setminus SC(X)$. Let $x \in X$ such that $g(x) \in D \setminus g(K)$ and either $x \in SC(X)$ or $g(x) \in Cn(D)$, and let $z \in I(K, x) \cap K$. Since z is not in $SC(X)$ and $g(x) \neq g(z)$, by Lemma 2.4, $int(Q_x(z)) = \emptyset$. So by Lemma 2.5, $Q_x(z)$ is a shore set. But $K \subset Q_x(z)$. So K is also a shore set in X. ☐

As noted in the introduction, if D is a dendroid with $SC(D) \neq \emptyset$, then $D \setminus S(D) = SC(D)$. A consequence of this result and Lemma 2.2 is that $SC(D)$ is arc connected. This was shown in [2] for dendroids, but that proof relied heavily on the arc connectedness of D. The result in [2] was a little stronger though. It also says that if x and y are strong centers in a dendroid D, then there are basins U and V such that each point on the arc xy is a strong center with the same basins U and V. It is not known if that is also true for λ-dendroids. We can prove the following.

Theorem 2.8. *If $g : X \to D$ is a monotone map from the λ-dendroid X onto the dendroid D such that $g(SC(X))$ contains more than one point, then $X \setminus S(X) = SC(X)$.*

Proof: Suppose $g : X \to D$ is a monotone map from the λ-dendroid X onto the dendroid D such that $g(SC(X))$ contains more than one point. Suppose $y \in X \setminus S(X)$. Then there is $x \in X$ such that $g(x) \neq g(y)$ and $g(x) \in SC(D)$. It follows from Lemma 2.5 that $int(Q_x(y)) \neq \emptyset$. Therefore, by Lemma 2.4, $y \in SC(X)$. ☐

Corollary 2.9. *Suppose X is a λ-dendroid such that $SC(X)$ intersects more than one stratum of X, then $SC(X)$ is λ-connected and $X \setminus S(X) = SC(X)$.*

Proof: This follows from Theorem 2.8 and Lemma 2.2. ☐

The next lemma gives us something a little stronger than the λ-connectedness of $SC(X)$ proven above. It also reveals a little more about how the structure of $SC(X)$ is related to $SC(D)$ when $g : X \to D$ is a monotone map from λ-dendroid X to dendroid D.

Lemma 2.10. *If $g : X \to D$ is a monotone map from the λ-dendroid X onto the dendroid D, and x and y are different elements of $SC(D)$, then $I(g^{-1}(x), g^{-1}(y)) \subset SC(X)$*

Proof: Suppose $g : X \to D$ is a monotone map from the λ-dendroid X onto the dendroid D, and x and y are elements of $SC(D)$ such that $x \neq y$. According to [2, Theorem 2], there are open sets U and V in D that are basins for both x and y. So, if K is a continuum in X such that $K \cap g^{-1}(U) \neq \emptyset$ and $K \cap g^{-1}(V) \neq \emptyset$, then $g(K)$ contains x and y. So $K \cap g^{-1}(x) \neq \emptyset$ and $K \cap g^{-1}(y) \neq \emptyset$. Thus, $g^{-1}(U)$ and $g^{-1}(V)$ are basins for each element of $I(g^{-1}(x), g^{-1}(y))$. $\square$

Note that we have not proven anything about centers in a λ-dendroid, only strong centers. In fact, it is true that most of the work in this paper succeeds by avoiding the topic of centers in a λ-dendroid.

Theorem 2.11. *If $g : X \to D$ is a monotone map from the λ-dendroid X onto the dendroid D such that $C(D)$ contains more than one point, then $SC(X) \neq \emptyset$, $g(SC(X))$ contains more than one point, and $X \setminus S(X) = SC(X)$.*

Proof: According to [2, Theorem 2], if x and y are two points in $C(D)$, then $xy \setminus \{x, y\} \subset SC(D)$. So if $C(D)$ contains more than one point, then $SC(D)$ contains more than one point. It follows from Lemma 2.10 that $g(SC(X))$ contains more than one point. Therefore, by Theorem 2.8, $X \setminus S(X) = SC(X)$. $\square$

A center stratum of X is one whose image under the minimal monotone map from X onto a dendroid D is a center in D. Theorem 2.11 is restated in the following theorem.

Theorem 2.12. *Suppose X is a λ-dendroid with more than one center stratum, then $SC(X) \neq \emptyset$, $SC(X)$ intersects more than one stratum of X, and $X \setminus S(X) = SC(X)$.*

Theorem 2.13. *Suppose X is a λ-dendroid. If $SC(X) \neq \emptyset$, then every stratum that does not intersect $SC(X)$ is a shore set, and if*

$SC(X) = \emptyset$, *then every stratum that is not the center stratum of X*
is a shore set.

Proof: Suppose y is an element of the λ-dendroid X, $y^* \cap SC(X) = \emptyset$, and either $SC(X) \neq \emptyset$ or y^* is not a center stratum. It is easy to check that the continuum y^* satisfies the conditions of Lemma 2.7. $\square$

3. FINITE UNIONS OF SHORE CONTINUA

Illanes proved that if x and y are different elements of a dendroid X, then $int(cl(Q_x(y)) \cap cl(Q_y(x))) = \emptyset$ [3, Theorem 2]. In the proof, it is assumed that $int(cl(Q_x(y)) \cap cl(Q_y(x)))$ contains an open set O. Then a sequence is constructed inductively using only the fact that every point of $O \cap Q_x(y)$ is an accumulation point of $O \cap Q_y(x)$ and every point of $O \cap Q_y(x)$ is an accumulation point of $O \cap Q_x(y)$. This sequence leads to a contradiction. Therefore, the proof of Theorem 2 in [3] is valid with the slightly weaker hypothesis that there do not exist non-empty sets $V_1 \subset Q_x(y)$ and $V_2 \subset Q_y(x)$ such that $V_1 \subset cl(V_2)$ and $V_2 \subset cl(V_1)$. This is stated in the following lemma.

Lemma 3.1. *If D is a dendroid and x and y are elements of D with $x \neq y$, then there do not exist non-empty sets $V_1 \subset Q_x(y)$ and $V_2 \subset Q_y(x)$ such that $V_1 \subset cl(V_2)$ and $V_2 \subset cl(V_1)$.*

Lemma 3.2. *If X is a λ-dendroid and x and y are elements of X such that $x^* \neq y^*$, then there do not exist nonempty sets $V_1 \subset Q_x(y^*)$ and $V_2 \subset Q_y(x^*)$ such that $V_1 \subset cl(V_2)$ and $V_2 \subset cl(V_1)$.*

Proof: Suppose X is a λ-dendroid, x and y are elements of X such that $x^* \neq y^*$, and there do exist nonempty sets $V_1 \subset Q_x(y^*)$ and $V_2 \subset Q_y(x^*)$ such that $V_1 \subset cl(V_2)$ and $V_2 \subset cl(V_1)$. Let $g : X \to D$ be the minimal monotone map from X onto a dendroid D. Then $g(x) \neq g(y)$. Also, if $z \in Q_x(y)$ and C is a continuum in D that contains $g(x)$ and $g(z)$, then $g^{-1}(C)$ is a continuum in X that contains x and z. So $y \in g^{-1}(C)$, and thus, $g(y) \in C$; that is, $g(z) \in Q_{g(x)}(g(y))$. Therefore, $g(V_1) \subset g(cl(V_2)) \cap Q_{g(x)}(g(y)) = cl(g(V_2)) \cap Q_{g(x)}(g(y))$, and similarly, $g(V_2) \subset g(cl(V_1)) \cap Q_{g(y)}(g(x)) = cl(g(V_1)) \cap Q_{g(x)}(g(y))$. This contradicts Lemma 3.1. $\square$

Lemma 3.3. *If X is a λ-dendroid and x and y are elements of X such that $x^* \neq y^*$, then $cl(Q_x(y^*)) \cap cl(Q_y(x^*))$ has empty interior.*

Proof: Suppose X is a λ-dendroid, x and y are elements of X such that $x^* \neq y^*$, and there is an open set U contained in $cl(Q_x(y^*)) \cap cl(Q_x(y^*))$. If $V_1 = U \cap Q_x(y^*)$ and $V_2 = U \cap Q_y(x^*)$, then $V_1 \subset Q_x(y^*)$, $V_2 \subset Q_y(x^*)$, $V_1 \subset cl(V_2)$, and $V_2 \subset cl(V_1)$. This contradicts Lemma 3.2. $\qquad\square$

Now that a generalization of [3, Theorem 6] to λ-dendroids has been established, a few technical lemmas are still needed in order to prove a union theorem for shore continua in a λ-dendroid.

Lemma 3.4. *If A and B are subcontinua of a λ-dendroid X, $a \in A$, $b \in B$, and $x \in X \setminus (A \cup B)$ such that $Q_x(A) \cap Q_x(B) = \emptyset$, then $Q_a(B) = Q_x(B)$ and $Q_b(A) = Q_x(A)$.*

Proof: Suppose A and B are subcontinua of a λ-dendroid X, $a \in A$, $b \in B$, and $x \in X \setminus (A \cup B)$ such that $Q_x(A) \cap Q_x(B) = \emptyset$. Then $a \in A \subset Q_b(A) \subset X \setminus Q_x(B)$. So $I(x,a) \cap B = \emptyset$.

If $z \in Q_a(B)$, then $I(z,a) \cap B \neq \emptyset$. But $I(z,a) \subset I(z,x) \cup I(x,a)$. So $I(z,x) \cap B \neq \emptyset$, and thus, $z \in Q_x(B)$. So $Q_a(B) \subset Q_x(B)$.

If $w \in Q_x(B)$, then $I(w,x) \cap B \neq \emptyset$. But $I(w,x) \subset I(w,a) \cup I(a,x)$. So $I(w,a) \cap B \neq \emptyset$, and thus, $w \in Q_a(B)$. So $Q_x(B) \subset Q_a(B)$. Therefore, $Q_x(B) = Q_a(B)$, and similarly, $Q_x(A) = Q_b(A)$. $\qquad\square$

Lemma 3.5. *If A and B are subcontinua of a λ-dendroid X such that $A^* \cap B^* = \emptyset$ and $x \in X \setminus A \cup B$ such that $Q_x(A) \cap Q_x(B) = \emptyset$, then $int(cl(Q_x(A)) \cap cl(Q_x(B))) = \emptyset$.*

Proof: Suppose A and B are subcontinua of a λ-dendroid X such that $A^* \cap B^* = \emptyset$, and $x \in X \setminus A \cup B$ such that $Q_x(A) \cap Q_x(B) = \emptyset$. Let $a \in I(A,B) \cap A$ and $b \in I(A,B) \cap B$. Then $a^* \cap b^* = \emptyset$. So, by Lemma 3.3, $int(cl(Q_a(b^*) \cap Ccl(Q_b(a^*))) = \emptyset$. Now, $Q_a(B) = Q_a(b) \subset Q_a(b^*)$ and similarly, $Q_b(A) \subset Q_b(a^*)$. So $int(cl(Q_b(A)) \cap cl(Q_a(B))) = \emptyset$. Finally, by Lemma 3.4, $Q_b(A) = Q_x(A)$ and $Q_a(B) = Q_x(B)$, so $int(cl(Q_x(A) \cap Q_x(B))) = \emptyset$. $\qquad\square$

Lemma 3.6. *If K_1 and K_2 are subcontinua of a λ-dendroid X such that $K_1{}^* \cap K_2{}^* = \emptyset$, and if $x \in X \setminus (K_1^* \cup K_2^*)$, then $Q_x(K_1) \cap Q_x(K_2) \neq \emptyset$ implies that either $Q_x(K_1) \subset Q_x(K_2)$ or $Q_x(K_2) \subset Q_x(K_1)$.*

Proof: Assume K_1 and K_2 are continua in a λ-dendroid X such that $K_1^* \cap K_2^* = \emptyset$ and $x \in X \setminus (K_1^* \cup K_2^*)$. Suppose that $w \in$

$Q_x(K_1) \cap Q_x(K_2)$, and, in order to get a contradiction, assume $y \in Q_x(K_1) \setminus Q_x(K_2)$ and $z \in Q_x(K_2) \setminus Q_x(K_1)$.

Note that $I(K_1, x) \cap Q_x(K_2) = \emptyset$ since $I(K_1, x) \subset I(y, x)$ and $I(y, x) \cap K_2 = \emptyset$. Similarly, $I(K_2, x) \cap Q_x(K_1) = \emptyset$.

Also, if $t \in I(K_1, x) \cap Q_x(K_1)$, then $I(K_1, x) \subset I(t, x)$. So $I(K_1, x) = I(t, x)$, and thus, $t \in K_1^*$ by Lemma 2.3. It follows that $I(K_1, x) \subset (X \setminus (Q_x(K_1) \cup Q_x(K_2)) \cup K_1^*$. Similarly, $I(K_2, x) \subset (X \setminus (Q_x(K_1) \cup Q_x(K_2)) \cup K_2^*$.

Let $L = K_1^* \cup I(K_1, x) \cup I(x, K_2) \cup K_2^*$; then $L \subset X \setminus (Q_x(K_1) \cup Q_x(K_2)) \cup K_1^* \cup K_2^*$.

If $t \in I(K_1, w) \setminus K_1^*$, then $I(t, x) \cap K_1 \neq \emptyset$ since $I(w, t) \cap K_1 = \emptyset$ and $I(w, x) \cap K_1 \neq \emptyset$. So $I(K_1, w,) \subset Q_x(K_1) \cup K_1^*$, and similarly, $I(K_2, w,) \subset Q_x(K_2) \cup K_2^*$. Let $M = K_1^* \cup I(K_1, w) \cup I(w, K_2) \cup K_2^*$. Then $M \subset Q_x(K_1) \cup Q_x(K_2) \cup K_1^* \cup K_2^*$.

But L and M are continua and $L \cap M = K_1^* \cup K_2^*$, which contradicts the fact that X is hereditarily unicoherent.   □

The following lemma is restated here since it will be used several times in what follows in this section and the next section.

Lemma 3.7. [7, Lemma 6] *If* $\{X_1, X_2, ..., X_n\}$ *is a finite collection of sets in a topological space such that the interior of each* X_i *is empty, and for* $i \neq j$ *the interior of* $\overline{X_i} \cap \overline{X_j}$ *is empty, then the interior of* $X_1 \cup X_2 \cup ... \cup X_n$ *is empty*

Theorem 3.8. *If* X *is a* λ*-dendroid such that* $SC(X) \neq \emptyset$ *and* $\{K_1, K_2, ..., K_n\}$ *is a finite collection of shore continua in* X *such that for* $i \neq j$, $K_i^* \cap K_j^* \cap SC(X) = \emptyset$, *then* $\bigcup K_i$ *is a shore set.*

Proof: Suppose X is a λ-dendroid such that $SC(X) \neq \emptyset$ and $\{K_1, K_2, ..., K_n\}$ is a finite collection of shore continua in X such that for $i \neq j$, $K_i^* \cap K_j^* \cap SC(X) = \emptyset$. Suppose for some $i \neq j$ that $K_i^* \cap K_j^* \neq \emptyset$. Then there is a $z \in X \setminus SC(X)^*$ such that $z^* \subset K_i^* \cap K_j^*$. It follows from Lemma 2.6 that $K_i \cup z^* \cup K_j$ is a shore set. If a new collection is formed by replacing K_i and K_j with $K_i \cup z^* \cup K_j$, this new collection will still satisfy the assumption that no two elements will intersect the same stratum of $SC(X)^*$. So, without loss of generality, assume $K_i^* \cap K_j^* = \emptyset$.

Let $x \in SC(X)$. We will show that $int(\cup Q_x(K_i)) = \emptyset$. By Lemma 3.6, if $Q_x(K_i) \cap Q_x(K_j) \neq \emptyset$, then $Q_x(K_i) \subset Q_x(K_j)$ or $Q_x(K_j) \subset Q_x(K_i)$. So without loss of generality assume $Q_x(K_i) \cap$

$Q_x(K_j) = \emptyset$. Thus, by Lemma 3.5, if $i \neq j$, then $int(cl(Q_x(K_i) \cap cl(Q_x(K_j)) = \emptyset$. It follows from Lemma 2.5 that $int(Q_x(K_i) = \emptyset$ for each i. So, by Lemma 3.7, $int(\cup Q_x(K_i)) = \emptyset$. Since $X \setminus \cup Q_x(K_i)$ is λ-connected, $\cup Q_x(K_i)$ is a shore set by Lemma 2.5, and therefore, $\cup (K_i)$ is a shore set. $\qquad\qquad\square$

4. The λ-components of $X \setminus SC(X)^*$

In this section, it will be shown that for a λ-dendroid X a finite union of the λ-components of $X \setminus SC(X)^*$ is a shore set. That implies that for a dendroid D, a finite union of arc components of $D \setminus SC(D)$ is shore set, which was proven in [7, Theorem 4]. One might wonder about the λ-components of $X \setminus SC(X)$. Here is the problem. The proof we have establishes that the complement of the union of a finite number of λ-components of $X \setminus SC(X)^*$ is dense in X and λ-connected. But the complement of even one λ-component of $X \setminus SC(X)$ may not be λ-connected. Consider the following example.

In the plane, let $A = \{(x, y) | y = sin(1/x), 0 < x \leq 1\}$. Let $\{L_i\}$ be a collection of arcs in the plane such that for $i \neq j$ $L_i \cap L_j = \{(0,0)\}$; such that for each i, $L_i \cap cl(A) = \{(0,0)\}$; and such that $lim(L_i) = cl(A)$. Let $X = \cup\{L_i\} \cup cl(A)$. Then X is clearly a λ-dendroid and $SC(X)$ is the union of the L_i minus their free endpoints. Let $B = \{(0, y) | 0 < y \leq 1\}$ and let $C = \{(0, y) | -1 \leq y < 0\}$. Clearly A, B, and C are each λ-components of $S(X) = X \setminus SC(X)$. Neither $X \setminus B$ nor $X \setminus C$ is λ-connected.

In the example above, it turns out that $SC(X)$ is dense and λ-connected, and so $S(X)$ is itself a shore set. It still might be true that the λ-components of $X \setminus SC(X)$ are shore sets or even that a finite union of the λ-components of $X \setminus SC(X)$ is a shore set.

Lemma 4.1. *If B is a nonempty λ-connected subset of the λ-dendroid X and C is a λ-component of $X \setminus B^*$, then $C \cup B^*$ and $X \setminus C$ are both λ-connected.*

Proof: Suppose B is a nonempty λ-connected subset of the λ-dendroid X and C is a λ-component of $X \setminus B^*$. Since B^* is also λ-connected, it suffices to show that if $y \in C$ and $x \in B^*$, then $I(y, x) \subset C \cup B^*$. So suppose $y \in C$ and $x \in B^*$ and $z \in I(y, x) \setminus B^*$. If there exists $w \in I(y, z) \cap B^*$, then $I(y, z) \subset I(y, x) \subset I(y, w) \cup$

$I(w,x)$. But $I(w,x) \subset B^*$. So $z \in I(y,w)$. But then $I(y,w) = I(y,z)$, which implies that $z^* = w^*$ by Lemma 2.3, and therefore, $z \in B^*$ contrary to our assumption. Thus, $I(y,z) \cap B^* = \emptyset$. So $I(y,z) \subset C$. Thus, every point of $I(y,x)$ that is not in B^* is contained in C.

Now $X \setminus C$ is the union of all λ-components of $X \setminus B^*$ not equal to C with B^*. So $X \setminus C$ is also λ-connected.   □

Lemma 4.2. *If B is a nonempty λ-connected subset of the λ-dendroid X, and C is a λ-component of $X \setminus B^*$, $y \in C$, $z \in C$, and $x \in X \setminus B^*$, then $I(y,x) \setminus C = I(z,x) \setminus C$.*

Proof: Suppose B is a nonempty λ-connected subset of the λ-dendroid X and C is a λ-component of $X \setminus B^*$, $y \in C$, $z \in C$, and $x \in X \setminus B^*$. Then $I(y,x) \subset I(y,z) \cup I(z,x)$ and therefore, $I(y,x) \setminus C \subset (I(y,z) \setminus C) \cup (I(z,x) \setminus C) = \emptyset \cup (I(z,x) \setminus C) = (I(z,x) \setminus C)$. Similarly $I(z,x) \setminus C \subset I(y,x) \setminus C$.   □

For B, a nonempty λ-connected subset of the λ-dendroid X, and λ-component C of $X \setminus B^*$, and $x \in B^*$, according to Lemma 4.2, we can define $J(C,x) = cl(I(y,x) \setminus C)$ where y is any element of C. It follows that $J(C,x)$ is a continuum that is contained in every continuum that intersects C and contains x. In fact, $J(C,x)$ is a maximal continuum with respect to being contained in every continuum that intersects C and contains x. A λ-component C of $X \setminus B^*$ will be called type I if there is an $x \in B^*$ such that $J(C,x) \cap C \neq \emptyset$. Otherwise, C will be called type II.

Lemma 4.3. *If B is a nonempty λ-connected subset of the λ-dendroid X, and C is a type II λ-component of $X \setminus B^*$, and $g : X \to D$ is a minimal monotone map of X onto a dendroid D, then there is a unique stratum z^* contained in B^* such that for each x in B^*, the set $g(J(C,x))$ is the arc in D from $g(z^*)$ to $g(x)$.*

Proof: Suppose B is a nonempty λ-connected subset of the λ-dendroid X, and C is a type II λ-component of $X \setminus B^*$, and $g : X \to D$ is a minimal monotone map of X onto a dendroid D. Choose $x \in B^*$ and $y \in C$. Then $g(J(C,x))$ is a subarc of the arc $g(I(y,x))$. So there is a $z \in J(C,x)$ such that $g(J(C,x))$ is the arc from $g(z)$ to $g(x)$. Since $J(C,x)$ does not depend on y, z does not depend on y. Also note that $J(C,x) \cap C = \emptyset$ and $z \in J(C,x) \subset I(y,x) \subset B^* \cup C$. So $z^* \subset B^*$.

Now, for any $y \in C$, it will be shown that for z chosen in this way, $I(y, z) \subset C \cup z^*$. Assume it is not the case and there exists $w \in I(y, z) \setminus (C \cup z^*)$. Note that z is not in $I(y, w)$, because if $z \in I(y, w)$, then $I(y, w) = I(y, z)$, and then, by Lemma 2.3, $w \in z^*$. But $B^* \cup C$ is λ-connected by Lemma 4.1. So $I(y, z) \subset B^* \cup C$, and therefore, $w \in B^* \setminus z^*$. Thus, $I(w, x) \subset B^*$, and therefore, $I(w, x) \subset I(y, x) \setminus C \subset J(C, x)$. So $g(w) \in g(J(C, x))$ and $g(J(C, x))$ is the arc in D from $g(z)$ to $g(x)$. On the other hand, $z \in I(w, x)$ since $z \in I(y, x) \subset I(y, w) \cup I(w, x)$, and z is not in $I(y, w)$. So $g(z)$ is contained in the arc from $g(w)$ to $g(x)$. So $g(w) = g(z)$. But, again, that makes $w \in z^*$, which is a contradiction. So $I(y, z) \subset C \cup z^*$ for each $y \in C$.

Now suppose x' is also a point in B^* and $z' \in J(C, x)$ is chosen such that $g(J(C, x))$ is the arc from $g(z')$ to $g(x)$. Let y be any element of C. Then $I(z, z') \subset I(y, z) \cup I(y, z') \subset C \cup z^* \cup z'^*$. And since B^* is λ-connected, $I(z, z') \subset B^*$. It follows that $I(z, z') \subset z^* \cup z'^*$. Thus, $z^* = z'^*$. $\square$

Lemma 4.4. *If B is a nonempty λ-connected subset of a λ-dendroid X, and $\{C_1, C_2, ..., C_n\}$ is a finite collection of λ-components of $X \setminus B^*$ such that $int(C_i) = \emptyset$ for each i, then $\bigcup C_i$ is a shore set.*

Proof: Suppose B is a nonempty λ-connected subset of a λ-dendroid X and $\{C_1, C_2, ..., C_n\}$ is a finite collection of λ-components of $X \setminus B^*$ such that $int(C_i) = \emptyset$ for each i. A stratum of X will be associated with each C_i as follows. If C_i is type I, then let x_i be any element of B^* such that $J(C_i, x_i) \cap C_i \neq \emptyset$ and let $z_i \in J(C_i, x_i) \cap C_i$. Then the stratum associated with C_i is z_i^*. Note that in this case $z_i^* \subset C_i$.

If C_i is type II, the stratum associated with C_i is unique stratum z_i^* from Lemma 4.3.

Note that it is possible that more than one type II element of $\{C_1, C_2, ..., C_n\}$ is associated with the same stratum, but this is not possible for type I elements.

Next, it will be shown that if $x \in X \setminus C_i$, then $C_i \subset Q_x(z_i^*)$.

Suppose C_i is type I, and $x \in X \setminus C_i$. Let $y \in C_i$, then $I(y, x)$ must contain some $v \in B^*$, and $I(y, v)$ must contain z_i because $z_i \in J(C_i, x_i) \subset I(y, x_i) \subset I(y, v) \cup I(v, x_i)$, and z_i is not in $I(v, x_i)$ since $I(v, x_i) \subset B^*$, and $z_i \in C_i$. Thus, $C_i \subset Q_x(z_i) \subset Q_x(z_i^*)$.

Now suppose C_i is type II. This implies that $z_i^* \cap J(C_i, x') \neq \emptyset$ for any $x' \in B^*$, and since for any $y \in C_i$, $J(C_i, x') \subset I(y, x')$, it follows that $z_i^* \cap I(y, x') \neq \emptyset$ for any $x' \in B^*$. Now, if $x \in X \setminus C_i$, and $y \in C_i$, then $I(y, x)$ must contain some $x' \in B^*$. So, $z_i^* \cap I(y, x) \supset z_i^* \cap I(y, x') \neq \emptyset$. We have shown for each $y \in C_i$ and each $x \in X \setminus C_i$ that $I(y, x) \cap z_i^* \neq \emptyset$. That is, $C_i \subset Q_x(z_i^*)$ for each $x \in X \setminus C_i$.

Let $\{W_1, W_2, ... W_r\}$ be a finite collection of sets in X such that for each $j \leq r$ there is a z_j' such that W_j is the union of all of the elements of $\{C_1, C_2, ..., C_n\}$ that are associated with $z_j'^*$. Note that if W_i contains a C_j that is type I, then $W_i = C_j$. Otherwise, W_i is a union of type II elements of $\{C_1, C_2, ..., C_n\}$. In either case, since $int(C_i) = \emptyset$ for each i, we have $int(W_i) = \emptyset$.

According to Lemma 3.7, if $int(W_i) = \emptyset$ for each i and $int(cl(W_i) \cap cl(W_j)) = \emptyset$ for $i \neq j$, then $\bigcup W_i$ has empty interior. From the preceding arguments we see that for each i if $x \in X \setminus W_i$, then $W_i \subset Q_x(z_i'^*)$. From Lemma 3.3, we have that if $i \neq j$, then $int(cl(W_i) \cap cl(W_j)) \subset int(cl(Q_{z_i'}(z_j'^*)) \cap cl(Q_{z_j'}(z_i'^*))) = \emptyset$.

Therefore, it follows from Lemma 3.7 that $\bigcup C_i = \bigcup W_i$ has empty interior. Since $X \setminus (\bigcup C_i)$ is clearly λ-connected, it follows from Lemma 2.5 that $\bigcup C_i$ is a shore set. $\square$

Theorem 4.5. *If X is a λ-dendroid such that $SC(X) \neq \emptyset$, then the union of a finite number of λ-components of $X \setminus SC(X)^*$ is a shore set. Also, if $SC(X)^*$ is closed, then the union of a countable number of λ-components of $X \setminus SC(X)^*$ is a shore set.*

Proof: Suppose X is a λ-dendroid such that $SC(X) \neq \emptyset$ and $\{C_1, C_2, \ldots, C_n\}$ is a finite collection of λ-components of $X \setminus SC(X)^*$.

If C_i is type I, then let x_i be any element of $SC(X)^*$ such that $J(C_i, x_i) \cap C_i \neq \emptyset$ and let $z_i \in J(C_i, x) \cap C_i$. Then z_i is a shore point, and if $x \in SC(X)$, then by Lemma 2.5 $int(Q_x(z_j)) = \emptyset$. Since $C_j \subset Q_x(z_j)$ it follows in this case that $int(C_j) = \emptyset$.

Now suppose C_j is type II. Fix $y \in C_j$. For each positive integer n, let $K_n = \{z \in C_j | I(y, z) \cap B_{\frac{1}{n}}(J(C_j, x_j)) = \emptyset\}$. Since K_n is a continuum in C_j, and since $K_n \subset X \setminus SC(X)$ by Lemma 2.7, K_n is a shore set. So $int(K_n) = \emptyset$. Since $C_j = \bigcup \{K_n\}$, C_j is a countable union of closed sets with empty interior. It follows from the Baire theorem that C_j has empty interior. So by Lemma 4.4, $\bigcup C_i$ is a shore set.

Finally, if $SC(X)^*$ is closed, then every λ-component of $X \setminus$ $SC(X)^*$ is type II, and therefore, every λ-component of $X \setminus SC(X)^*$ is the union of a countable number of closed sets with empty interior. Thus, the countable union of λ-components of $X \setminus SC(X)^*$ has empty interior and has λ-connected complement and is, therefore, a shore set. $\qquad\square$

One nice feature of this classification of the points of a λ-dendroid into shore points and strong centers is that when there are no strong centers or when the strong centers are contained in one stratum, we seem to know even more about the shore points and shore sets. In the case of a dendroid D, this is because there can be only one center x and $D \setminus \{x\}$ has to have uncountably many arc components and no finite union of these has nonempty interior. The previous theorem establishes similar facts for λ-dendroids. Note that if the strong centers are contained in a single stratum, then $SC(X)^*$ is closed, and therefore, if $X \setminus SC(X)^* \neq \emptyset$, then $X \setminus SC(X)^*$ has uncountably many λ-components, and a finite union of these has empty interior. The next theorem says the same thing for the case where there are no strong centers.

Theorem 4.6. *If X is a λ-dendroid such that $SC(X) = \emptyset$, M is the center stratum of X, and $X \setminus M \neq \emptyset$, then $X \setminus M$ has uncountably many λ-components, and any set that is the union of a countable number of λ-components of $X \setminus M$ is a shore set.*

Proof: Assume X is a λ-dendroid such that $SC(X) = \emptyset$, and M is the center stratum of X. Suppose C is a λ-component of $X \setminus M$ and fix $y \in C$. For each positive integer n, let $K_n = \{z \in C | I(y, z) \cap B_{\frac{1}{n}}(M) = \emptyset\}$. Each K_n is a continuum in C and if x is any point in M, then K_n and x satisfy the conditions of Lemma 2.7. So K_n is a shore set.

So $int(K_n) = \emptyset$. Clearly $C = \cup K_n$. So each λ-component of $X \setminus M$ is the countable union of continua with empty interior. Therefore, the union of countably many λ-components of $X \setminus M$ has empty interior. By Lemma 4.1, the complement of the union any number of λ-components of $X \setminus M$ is a λ-connected set. It follows that the union of countably many λ-components of $X \setminus M$ is a shore set since it has a dense λ-connected complement in X.

If $X \setminus M$ has only countably many λ-components, then their union is a shore set, so M must be dense in X. But M is a continuum so $M = X$. $\qquad\square$

Theorem 4.6 also says that if $X \setminus M$ has only countably many λ-components, then M is the only stratum of X. Charatonik has constructed several examples of λ-dendroids with only one stratum, but each of those also contains strong centers. It may be possible to construct a λ-dendroid with only one stratum and no strong centers. It would be a very complicated λ-dendroid.

We end this discussion with two questions.

Question 4.7. *In a λ-dendroid, is the union of two disjoint shore continua a shore set?*

Note that the only case left to answer for the previous question is when A and B are shore continua in the λ-dendroid X, and there is an $x \in X$ such that $A \cap x^* \neq \emptyset$ and $B \cap x^* \neq \emptyset$.

Question 4.8. *In a λ-dendroid X, are the λ-components of $X \setminus SC(X)$ shore sets?*

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