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Boundary Values in Range Spaces of Co-Analytic Truncated Toeplitz Operator

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BOUNDARY VALUES IN RANGE SPACES OF CO-ANALYTIC TRUNCATED TOEPLITZ OPERATORS

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ABSTRACT. Functions in backward shift invariant subspaces have nice analytic continuation properties outside the spectrum of the inner function defining the space. Inside the spectrum of the inner function, Ahern and Clark showed that under some distribution condition on the zeros and the singular measure of the inner function, it is possible to obtain non-tangential boundary values of every function in the backward shift invariant subspace as well as for their derivatives up to a certain order. Here we will investigate, at least when the inner function is a Blaschke product, the non-tangential boundary values of the functions of the backward shift invariant subspace after having applied a co-analytic (truncated) Toeplitz operator. There appears to be a smoothing effect.

1. INTRODUCTION

Let H^2 denote the Hardy space of the open unit disk $\mathbb{D} = \{|z| < 1\}$ and $L^2 = L^2(d\theta/2\pi)$ denote the classical Lebesgue space of the unit circle $\mathbb{T} = \{|z| = 1\}$ with norm $\|\cdot\|$. H^2 is regarded as a closed subspace of L^2 in the usual way via non-tangential boundary values. For an inner function I, we let $K_I = H^2 \ominus IH^2$ be the well-known model space [Nik86].

The boundary behavior of functions in K_I have been well studied. For example, every function in K_I has a meromorphic pseudo-continuation to the extended exterior disk [CR00, DSS70, RS02]: For every $f \in K_I$, there is a meromorphic function F on the extended exterior disk whose non-tangential boundary values match those of f almost everywhere. As another example [Moe62], every $f \in K_I$ has an analytic continuation across $\mathbb{T} \setminus \sigma(I)$, where

$$\sigma(I) = \left\{ |z| \le 1 : \varliminf_{\lambda \to z} |I(\lambda)| = 0 \right\}$$

is the spectrum of I. If $I = B_\Lambda s_\mu$, where B_Λ is the Blaschke factor with zeros $\Lambda = \{\lambda_n\}_{n\ge 1} \subset \mathbb{D}$ (repeated according to multiplicity) and s_μ is the singular inner factor with associated singular measure μ on $\mathbb{T}$, then

$$\sigma(I) = \Lambda^- \cup \operatorname{suppt}(\mu).$$

Note that every function in K_I has a pseudo-continuation across $\mathbb{T}$ although, if the Blaschke product has zeros which accumulate everywhere on $\mathbb{T}$ or if the support of μ is all of $\mathbb{T}$, for example, functions in K_I might not have an analytic continuation across any subarc of $\mathbb{T}$.

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Our starting point for this paper is a result of Ahern and Clark [AC70b] which examines the non-tangential boundary behavior of functions in K_I even closer by considering what happens near $\sigma(I)$ where analytic continuation is not guaranteed. To state their result, we set a bit of notation: Let P_I be the orthogonal projection of L^2 onto K_I and $A_z : K_I \to K_I, A_z f = P_I(zf)$ be the compression of the shift ('multiplication by z' on H^2) to K_I.

Theorem 1.1 ([AC70b])**.** *For an inner function $I = B_\Lambda s_\mu$ and $\zeta \in \mathbb{T}$, the following are equivalent:*

(1) *Every $f \in K_I$ has a non-tangential limit at ζ, i.e.,*

$$f(\zeta) := \angle \lim_{\lambda \to \zeta} f(\lambda)$$

exists.

(2) *For every $f \in K_I$, $f(\lambda)$ is bounded as $\lambda \to \zeta$ non-tangentially.*

(3) $P_I 1 \in \operatorname{Rng}(\operatorname{Id} - \overline{\zeta} A_z)$.

(4) $(\operatorname{Id} - \overline{\lambda} A_z)^{-1} P_I 1$ *is norm bounded as $\lambda \to \zeta$ non-tangentially.*

(5) *I has an angular derivative in the sense of Caratheodory at ζ, i.e.,*

$$\angle \lim_{\lambda \to \zeta} I(\lambda) = \eta \in \mathbb{T}$$

and

$$\angle \lim_{\lambda \to \zeta} I'(\lambda) \quad \textit{exists}.$$

(6) *The following two conditions hold:*

$$\sum_{n \geq 1} \frac{1 - |\lambda_n|^2}{|\zeta - \lambda_n|^2} < \infty \tag{1.2}$$

$$\int_{\mathbb{T}} \frac{d\mu(\xi)}{|\xi - \zeta|^2} < \infty. \tag{1.3}$$

This is only a partial statement of the Ahern-Clark result. They went on further to characterize the existence of non-tangential boundary limits of the derivatives (up to a given order) of functions in K_I.

Note that simple examples show that one can have an inner function I and a $\zeta \in \mathbb{T}$ such that every function in K_I has a non-tangential limit at ζ without necessarily having an analytic continuation to a neighborhood of ζ.

If one (and hence all) of the equivalent conditions of the Ahern-Clark theorem is satisfied, then it makes sense to evaluate functions $f \in K_I$ at ζ, and the corresponding point evaluation functional can be represented by k_ζ^I. That is to say that

$$f(\zeta) = \langle f, k_\zeta^I \rangle \quad \forall f \in K_I.$$

In this paper, we study the boundary values of functions in K_I even further - beyond pseudo-continuation, analytic continuation, or the above Ahern-Clark result - by replacing the function $P_I 1$ in conditions (3) and (4) in the Ahern-Clark theorem with $P_I h$ where $h \in H^2$.

Let us take a closer look at $(\mathrm{Id} - \overline{\lambda} A_z)^{-1} P_I 1$ from condition (4). Since $(A_z)^n g = P_I z^n g$ for any $g \in K_I$, we get, for every $f \in K_I$ and $\lambda \in \mathbb{D}$,

$$\begin{aligned} \langle f, (\mathrm{Id} - \overline{\lambda} A_z)^{-1} P_I 1 \rangle &= \left\langle f, \sum_{n=0}^{\infty} \overline{\lambda}^n (A_z)^n P_I 1 \right\rangle = \left\langle f, \sum_{n=0}^{\infty} \overline{\lambda}^n z^n P_I 1 \right\rangle = \left\langle f, \frac{1}{1 - \overline{\lambda} z} P_I 1 \right\rangle \\ &= \left\langle f, \frac{1}{1 - \overline{\lambda} z} (P_I 1 - 1) + \frac{1}{1 - \overline{\lambda} z} 1 \right\rangle = \left\langle f, \frac{1}{1 - \overline{\lambda} z} 1 \right\rangle \\ &= f(\lambda). \end{aligned} \tag{1.4}$$

Thus $(\mathrm{Id} - \overline{\lambda} A_z)^{-1} P_I 1$ is the reproducing kernel k_λ^I for the model space K_I and the Ahern-Clark theorem gives a condition as to when k_λ^I converges weakly to the boundary reproducing kernel function k_ζ^I as $\lambda \to \zeta$ non-tangentially.

When $P_I 1$ is replaced by $P_I h$, where $h \in H^2$, an analogous calculation to the one in (1.4) gives us, at least formally,

$$\langle f, (\mathrm{Id} - \overline{\lambda} A_z)^{-1} P_I h \rangle = (A_{\overline{h}} f)(\lambda), \tag{1.5}$$

where $A_{\overline{h}} f = P_I(\overline{h} f)$ is the truncated Toeplitz operator on K_I - which we assume to be bounded. Note that $A_{\overline{h}}$ is initially densely defined on bounded functions on K_I and, for certain h, can be extended to be bounded on K_I. Certainly if $h \in H^\infty$, the bounded analytic functions on $\mathbb{D}$, then $A_{\overline{h}}$ is bounded on K_I. However, there are unbounded $h \in H^2$ which yield bounded $A_{\overline{h}}$. We will discuss these details further in the next section. Truncated Toeplitz operators have been studied quite a lot recently and we refer the reader to the seminal paper by Sarason which started it all [Sar07].

By examining the weak convergence of the kernel functions

$$k_\lambda^h := (\mathrm{Id} - \overline{\lambda} A_z)^{-1} P_I h \tag{1.6}$$

as $\lambda \to \zeta$ (non-tangentially), we will determine the boundary behavior of functions in $\operatorname{Rng} A_{\overline{h}}$, the range of the truncated Toeplitz operator $A_{\overline{h}}$. Since $\operatorname{Rng} A_{\overline{h}} \subset K_I$, functions in this range will have finite non-tangential limits at at all points $\zeta \in \mathbb{T}$ where conditions (1.2) and (1.3) are satisfied. Certain choices of h can force other points $\zeta \in \mathbb{T}$ to be points of finite non-tangential limits. In the Section 5 of this paper, we will make a few remarks about the boundary behavior of the functions

$$f_h(\lambda) := \langle f, (\mathrm{Id} - \overline{\lambda} A_z)^{-1} P_I h \rangle$$

(which is the left-hand side of (1.5)), where the truncated Toeplitz operator $A_{\overline{h}}$ is not necessarily bounded and $f \in K_I$ is not necessarily in the domain of $A_{\overline{h}}$.

To state our main theorem, we introduce some notation. For $\lambda \in \mathbb{D}$, let

$$b_\lambda(z) = \frac{z - \lambda}{1 - \overline{\lambda} z}$$

be the single Blaschke factor with zero at λ. For a Blaschke product $B_\Lambda = \prod_{\lambda \in \Lambda} (|\lambda|/\lambda) b_\lambda$ with zeros $\Lambda = \{\lambda_n\}_{n \geq 1}$, repeated accordingly to multiplicity, let the Takenaka-Malquist-Walsh functions be defined by

$$\gamma_n(z) = \frac{\sqrt{1 - |\lambda_n|^2}}{1 - \overline{\lambda_n} z} \prod_{k=1}^{n-1} b_{\lambda_k}(z).$$

It is well known [Nik86, p. 117] [Tak25] that $\{\gamma_n : n \in \mathbb{N}\}$ is an orthonormal basis for K_{B_Λ}. In fact, this basis was used in the proof of the Ahern-Clark theorem mentioned earlier. With our notation set, our main result reads as follows.

Theorem 1.7. *When I is a Blaschke product with zeros $\Lambda = \{\lambda_n\}_{n\geq 1}$ and $h \in H^2$ so that $A_{\overline{h}}$ is bounded on K_I, every function in $\operatorname{Rng} A_{\overline{h}}$ has a finite non-tangential limit at $\zeta \in \mathbb{T}$ if and only if*

$$\sum_{n\geq 1} |(A_{\overline{h}}\gamma_n)(\zeta)|^2 < \infty. \tag{1.8}$$

The alert reader might question whether or not $(A_{\overline{h}}\gamma_n)(\zeta)$ in (1.8) actually exists. It is after all the non-tangential boundary value of a function from K_I. However, as we will see in the proof of this theorem, $A_{\overline{h}}\gamma_n$ will turn out to be a rational function whose poles lie outside of $\mathbb{D}^-$ and so $A_{\overline{h}}\gamma_n$ can be evaluated at ζ without any difficulty. Also observe that when $h = 1$,

$$\sum_{n\geq 1} |(A_{\overline{h}}\gamma_n)(\zeta)|^2 = \sum_{n\geq 1} |\gamma_n(\zeta)|^2 = \sum_{n\geq 1} \frac{1-|\lambda_n|^2}{|\zeta-\lambda_n|^2},$$

giving us condition (1.2) in the Ahern-Clark theorem.

The proof of Theorem 1.7 will show that when condition (1.8) is satisfied then, as $\lambda \to \zeta$ non-tangentially, the kernel functions k_λ^h from (1.6) converge weakly to some function $k_\zeta^h \in K_I$. This function turns out to be sort of a reproducing kernel for $\operatorname{Rng} A_{\overline{h}}$ at ζ in that

$$(A_{\overline{h}}f)(\zeta) = \langle f, k_\zeta^h\rangle \quad \forall\, f \in K_I.$$

We will see from the proof of Theorem 1.7 that

$$\|k_\zeta^h\|^2 = \sum_{n\geq 1} |(A_{\overline{h}}\gamma_n)(\zeta)|^2.$$

In Section 3 we will compute an explicit formula for $A_{\overline{h}}\gamma_n(\zeta)$ which turns out to be quite cumbersome in the general case. Still, we are able to give some examples in Section 4 of when the condition in (1.8) holds. We mention that when I is an interpolating Blaschke product [Gar07, Ch. VII], the condition in (1.8) becomes much simpler.

Theorem 1.9. *If I is an interpolating Blaschke product with zeros $\Lambda = \{\lambda_n\}_{n\geq 1}$ and $h \in H^2$ such that $A_{\overline{h}}$ is bounded on K_I, then every function in $\operatorname{Rng} A_{\overline{h}}$ has a finite non-tangential limit at $\zeta \in \mathbb{T}$ if and only if*

$$\sum_{n\geq 1} (1-|\lambda_n|^2)\left|\frac{h(\lambda_n)}{\zeta-\lambda_n}\right|^2 < \infty. \tag{1.10}$$

We will discuss an example in Section 4 which will show that this condition does, in general, not apply to non-interpolating Blaschke products. In fact, it already fails when we take a Blaschke product associated with a non-separated union of two interpolating sequences. Although we do not develop this further here, the corresponding example will show how one can obtain a condition for finite unions of interpolating Blaschke products.

Non-tangential boundary values of functions in spaces related to backward shift invariant subspaces have been studied recently. We would like to mention in particular the results by Fricain and Mashreghi dealing with de Branges-Rovnyak spaces $\mathcal{H}(b)$ [FM08a, FM08b] which are one

way of generalizing the backward shift invariant subspaces. See Sarason's book [Sar94] for relations between the spaces $\mathcal{M}(\overline{a}) := T_{\overline{a}}H^2$ and $\mathcal{H}(b)$ when b is non extreme (this guarantees that there is $a \in \mathrm{Ball}(H^\infty)$ such that $|a|^2 + |b|^2 = 1$). Our situation is somewhat different since we consider Toeplitz operators not on the whole H^2 but only on the model space K_I.

Finally, the first mentioned author has considered analytic continuation questions in weighted backward shift invariant subspaces which appear naturally in the context of kernels of Toeplitz operators [Har]. We refer the reader to the survey [FH10] for more information.

The reader has probably noticed that we only discuss inner functions I which are Blaschke products, i.e., I has no singular inner factor. We will make some comments at the end of the paper as to the difficulties which arise in the the presence of a singular inner factor.

A final word concerning numbering in this paper: in each section, we have numbered theorems, propositions, lemmas, corollaries **and** equations consecutively.

2. Preliminaries

For an inner function I, let $K_I = H^2 \ominus IH^2$ be the model space corresponding to I. Since H^2 is a reproducing kernel Hilbert space with kernel

$$k_\lambda(z) = \frac{1}{1-\overline{\lambda}z},$$

then so is K_I with reproducing kernel

$$k_\lambda^I(z) = (P_I k_\lambda)(z) = \frac{1-\overline{I(\lambda)}I(z)}{1-\overline{\lambda}z},$$

where P_I is the orthogonal projection of L^2 onto K_I. Note that these kernels are bounded functions and finite linear combinations of them form a dense subset of K_I. This enables us, for $\varphi \in L^2$, to define the operator A_φ densely on K_I by $A_\varphi f = P_I(\varphi f)$. These operators, called *truncated Toeplitz operators*, have many interesting properties [Sar07] which we won't get into here. We do, however, mention a few of them which will be important for our purposes.

First we note that the symbols which represent truncated Toeplitz operators are not unique. In fact [Sar07, Thm. 3.1]

$$A_{\varphi_1} = A_{\varphi_2} \Leftrightarrow \varphi_1 - \varphi_2 \in IH^2 + \overline{IH^2}. \tag{2.1}$$

Secondly, when φ is a bounded function then certainly the truncated Toeplitz operator A_φ extends to be a bounded operator on K_I with $\|A_\varphi\| \le \|\varphi\|_\infty$. However, there are bounded truncated Toeplitz operators (i.e., ones which extend to be bounded on K_I) which do not have a bounded symbol [BCF+10].

In this paper, we focus our attention on the co-analytic truncated Toeplitz operator $A_{\overline{h}}$, where $h \in H^2$. As mentioned earlier, when $h \in H^\infty$, the bounded analytic functions on $\mathbb{D}$, then $A_{\overline{h}}$ is bounded on K_I. Although by using (2.1) every bounded $A_{\overline{h}}$ has an unbounded symbol, a well-known result of Sarason [Sar67] says that if a co-analytic truncated Toeplitz operator is bounded, then it can be represented by a *bounded* co-analytic symbol.

The central step in the Ahern-Clark approach is to express the reproducing kernel k_λ^I in terms of the resolvent of a certain operator in λ applied to a fixed function:

$$k_\lambda^I = (\mathrm{Id} - \overline{\lambda} A_z)^{-1} P_I 1.$$

In this situation, the following lemma allows to deduce the existence of the boundary limits at a point $\zeta \in \mathbb{T}$ from the fact that $(\mathrm{Id} - \overline{\zeta} A_z)$ is injective and $P_I 1$ is in the range of this operator.

Lemma 2.2 ([AC70b])**.** *Let $\xi \in \mathbb{T}$ and L be a contraction on a Hilbert space $\mathcal{H}$ such that $(\mathrm{Id} - \xi L)$ is injective. Furthermore, let $\{\lambda_n\}_{n\geq 1}$ be sequence of points in $\mathbb{D}$ tending non-tangentially to ξ as $n \to \infty$. Then, for a fixed $y \in \mathcal{H}$, the sequence*

$$w_n = (\mathrm{Id} - \lambda_n L)^{-1} y$$

is uniformly bounded if and only if y belongs to the range of $(\mathrm{Id} - \xi L)$, in which case, w_n tends weakly to $w_0 = (\mathrm{Id} - \xi L)^{-1} y$.

Remark 2.3. Below we will apply this lemma to the operator A_z on K_I. Clearly A_z is a contraction on K_I. To show that $(\mathrm{Id} - \xi A_z)$ is injective, observe, for $f \in K_I$, that

$$(\mathrm{Id} - \xi A_z) f = 0 \Leftrightarrow P_I((1 - \xi z) f) = 0 \Leftrightarrow (1 - \xi z) f \in I H^2.$$

But since $z \mapsto (1 - \xi z)$ is an outer function, then I divides the inner part of f from which we get $f \in I H^2$ and so, since $f \in K_I = H^2 \ominus I H^2$, $f \equiv 0$.

As mentioned in (1.5), for $h \in H^2$, the function

$$k_\lambda^h = (\mathrm{Id} - \overline{\lambda} A_z)^{-1} P_I h$$

serves as a reproducing kernel for $\mathrm{Rng}\, A_{\overline{h}}$ in the sense that

$$(A_{\overline{h}} f)(\lambda) = \langle f, k_\lambda^h \rangle, \quad f \in K_I. \tag{2.4}$$

From this and the identity

$$(A_{\overline{h}} f)(\lambda) = \langle P_I(\overline{h} f), k_\lambda \rangle = \langle f, h k_\lambda^I \rangle = \langle f, P_I(h k_\lambda^I) \rangle, \quad \forall f \in K_I,$$

we also deduce that

$$k_\lambda^h = P_I(h k_\lambda^I). \tag{2.5}$$

The next proposition, similar to Theorem 1.1, begins to get at the boundary behavior of functions in $\mathrm{Rng}\, A_{\overline{h}}$. The proof is pretty much the same but we include it anyway for the sake of completeness.

Proposition 2.6. *For an inner function I, a point $\zeta \in \mathbb{T}$, and a function $h \in H^2$ so that $A_{\overline{h}}$ is bounded on K_I, the following are equivalent:*

(1) *Every function in $\mathrm{Rng}\, A_{\overline{h}}$ has a finite non-tangential limit at ζ.*
(2) *$P_I h \in \mathrm{Rng}(\mathrm{Id} - \overline{\zeta} A_z)$.*
(3) *k_λ^h is norm bounded as $\lambda \to \zeta$ non-tangentially.*

Proof. By (2.4), along with the uniform boundedness principle, we have (1) implies (3). Statement (3) is equivalent to (2) by Lemma 2.2. Statement (3) implies (1) follows from Lemma 2.2 and (2.4). ∎

Corollary 2.7. *The following statements are equivalent:*

(1) *Every function in* $\operatorname{Rng} A_{\overline{h}}$ *has a finite non-tangential limit at* ζ.
(2) *There exists* $u \in H^2$ *and* $k \in K_I$ *which solve the following interpolation problem*

$$P_I h = (1 - \overline{\zeta} z)k + Iu. \tag{2.8}$$

Proof. Assuming statement (1) holds, we can use Proposition 2.6 along with Lemma 2.2 to say that k_λ^h converges weakly to some $k_\zeta^h \in K_I$ as $\lambda \to \zeta$ non-tangentially and moreover,

$$k_\zeta^h = (\operatorname{Id} - \overline{\zeta} A_z)^{-1} P_I h.$$

Using the the general observation $P_I(zv) - zv = (P_I - \operatorname{Id})(zv) \in \operatorname{Ker} P_I = IH^2$ we see that

$$\begin{aligned} P_I h &= (\operatorname{Id} - \overline{\zeta} A_z) k_\zeta^h \\ &= k_\zeta^h - \overline{\zeta} A_z k_\zeta^h \\ &= k_\zeta^h - \overline{\zeta} z k_\zeta^h + Iu, \quad u \in H^2 \\ &= (1 - \overline{\zeta} z) k_\zeta^h + Iu. \end{aligned}$$

This shows that (1) implies (2). To show (2) implies (1), simply reverse the argument. ∎

The above proof also says the following.

Corollary 2.9. *If* $A_{\overline{h}} f$ *has a finite non-tangential limit at* ζ *for every* $f \in K_I$ *then*

$$(A_{\overline{h}} f)(\zeta) = \langle f, k_\zeta^h \rangle.$$

Proof. In this situation, using (2.5), we will have, for every $f \in K_I$,

$$\begin{aligned} \langle f, k_\zeta^h \rangle &= \angle \lim_{\lambda \to \zeta} \langle f, k_\lambda^h \rangle = \angle \lim_{\lambda \to \zeta} \langle f, P_I(h k_\lambda) \rangle = \angle \lim_{\lambda \to \zeta} \langle f, h k_\lambda \rangle = \angle \lim_{\lambda \to \zeta} \langle \overline{h} f, k_\lambda \rangle \\ &= (A_{\overline{h}} f)(\zeta). \end{aligned}$$

∎

3. The main results

Remark 3.1. Until we say otherwise, we will assume that $h \in H^2$ is chosen so that $A_{\overline{h}}$ is bounded on K_I. Furthermore, by (2.1), $A_{\overline{h}} = A_{\overline{P_I h}}$ and so we will also assume that $h \in K_I$.

We will proceed as in [AC70b]. For a Blaschke product I with zero set $\Lambda = \{\lambda_n\}_{n \geq 1}$, we have already introduced the functions

$$\gamma_n(z) = \frac{\sqrt{1 - |\lambda_n|^2}}{1 - \overline{\lambda} z} \underbrace{\prod_{k=1}^{n-1} b_{\lambda_k}(z)}_{=: B_{n-1}(z)}$$

which form an orthonormal basis for K_I.

It turns out that the central point in the result is the behavior of $A_{\overline{h}} \gamma_n$ at a boundary point. This is what we will determine now. Before proceeding though, we should justify that the expression $(A_{\overline{h}} \gamma_n)(\zeta)$ is always defined. First observe that γ_n belongs to K_{B_n}, a finite dimensional subspace of rational functions whose poles lie outside $\mathbb{D}^-$. Moreover, $A_{\overline{h}} \gamma_n \in K_{B_n}$. This is because

$A_{\overline{h}}$ acts on K_{B_n} as the restriction of the co-analytic Toeplitz operator $T_{\overline{h}}$, and $T_{\overline{h}}K_{B_n} \subset K_{B_n}$. Consequently, we can evaluate $A_{\overline{h}}\gamma_n$ at $\zeta \in \mathbb{T}$ without any difficulty.

Proposition 3.2. *let Λ be a Blaschke sequence and $h \in H^2$. Then, writing*

$$\prod_{l=1}^{n}(z-\lambda_l) = \prod_{l=1}^{r}(z-\mu_l)^{k_l},$$

we have, for any $\zeta \in \mathbb{T}$,

$$(A_{\overline{h}}\gamma_n)(\zeta) = \sqrt{1-|\lambda_n|^2}\sum_{l=1}^{r}\frac{1}{(k_l-1)!}\overline{\frac{d^{k_l-1}}{d\mu_l^{k_l-1}}\left[\frac{h(\mu_l)\prod_{m=1}^{n-1}(1-\overline{\lambda_m}\mu_l)}{(1-\overline{\zeta}\mu_l)\prod_{j=1,j\neq l}^{r}(\mu_l-\mu_j)^{k_j}}\right]}. \tag{3.3}$$

Proof. Since $A_{\overline{h}} = T_{\overline{h}}|K_I$ and $T_{\overline{h}}K_{B_n} \subset K_{B_n}$ we get, for $\lambda \in \mathbb{D}$,

$$(A_{\overline{h}}\gamma_n)(\lambda) = (T_{\overline{h}}\gamma_n)(\lambda) = (P_+\overline{h}\gamma_n)(\lambda).$$

This last quantity is now equal to

$$\langle \overline{h}\gamma_n, k_\lambda\rangle = \sqrt{1-|\lambda_n|^2}\langle k_{\lambda_n}B_{n-1}, hk_\lambda\rangle.$$

We thus have to compute

$$\langle k_{\lambda_n}B_{n-1}, hk_\lambda\rangle = \int_{\mathbb{T}}\frac{1}{1-\overline{\lambda}_n z}\prod_{l=1}^{n-1}\frac{z-\lambda_l}{1-\overline{\lambda_l}z}\overline{h(z)}\frac{1}{1-\lambda\overline{z}}dm(z).$$

Passing to the conjugate expression and then replacing the measure $dm = d\theta/(2\pi)$ by $dz/(2\pi iz)$ we get

$$\begin{aligned}
\overline{\langle k_{\lambda_n}B_{n-1}, hk_\lambda\rangle} &= \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{1}{1-\overline{z}\lambda_n}\prod_{l=1}^{n-1}\frac{1-\overline{\lambda_l}z}{z-\lambda_l}h(z)\frac{1}{1-\overline{\lambda}z}\frac{dz}{z}\\
&= \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{1}{z-\lambda_n}\prod_{l=1}^{n-1}\frac{1-\overline{\lambda_l}z}{z-\lambda_l}h(z)\frac{1}{1-\overline{\lambda}z}dz\\
&= \frac{1}{2\pi i}\int_{\mathbb{T}}\prod_{l=1}^{n}\frac{1}{z-\lambda_l}\left[\frac{h(z)\prod_{j=1}^{n-1}(1-\overline{\lambda_j}z)}{1-\overline{\lambda}z}\right]dz. \qquad (3.4)
\end{aligned}$$

Now let $\prod_{l=1}^{n}(z-\lambda_l) = \prod_{l=1}^{r}(z-\mu_l)^{k_l}$ where μ_l are the *different* zeros of B_n and k_l are their corresponding multiplicities. Then from the residue theorem we obtain:

$$\overline{\langle k_{\lambda_n}B_{n-1}, hk_\lambda\rangle} = \sum_{l=1}^{r}\frac{1}{(k_l-1)!}\frac{d^{k_l-1}}{d\mu_l^{k_l-1}}\left[\frac{h(\mu_l)\prod_{m=1}^{n-1}(1-\overline{\lambda_m}\mu_l)}{(1-\overline{\lambda}\mu_l)\prod_{j=1,j\neq l}^{r}(\mu_l-\mu_j)^{k_j}}\right].$$

This expression is perfectly well behaved for $\lambda \longrightarrow \zeta$, so that by conjugating back and multiplying by the normalization constant $\sqrt{1-|\lambda_n|^2}$, we obtain the desired result. ∎

In the situation of simple zeros we get a much nicer formula that we will use in the example at the end of this paper.

Corollary 3.5. *Let Λ be a Blaschke sequence with simple zeros. Then we have, for each $\zeta \in \mathbb{T}$,*

$$(A_{\overline{h}}\gamma_n)(\zeta) = \sqrt{1-|\lambda_n|^2}\sum_{l=1}^{n}\frac{\overline{h(\lambda_l)}}{1-\overline{\lambda_l}\zeta}\frac{1}{\overline{(B_n)_{\lambda_l}(\lambda_l)}}\frac{1-|\lambda_l|^2}{1-\overline{\lambda_l}\lambda_n}$$

where

$$(B_n)_{\lambda_l} = \prod_{k=1,k\neq l}^{n} b_{\lambda_k}.$$

The interesting observation here is that the expression $|(B_n)_{\lambda_l}(\lambda_l)|$ measures, in a sense, the deviation of Λ from an interpolating sequence. This will be very useful in our Example 4.9.

Proof. Starting from the computation (3.4) the residue theorem now gives:

$$\overline{\langle k_{\lambda_n} B_{n-1}, hk_\lambda\rangle} = \sum_{l=1}^{n} \frac{h(\lambda_l)\prod_{j=1}^{n-1}(1-\overline{\lambda_j}\lambda_l)}{1-\overline{\lambda}\lambda_l} \prod_{j=1,j\neq l}^{n} \frac{1}{\lambda_l-\lambda_j}.$$

We split the above sum in two pieces $l \le n-1$ and $l = n$ and do some regrouping to get

$$\begin{aligned}&\sum_{l=1}^{n} \frac{h(\lambda_l)\prod_{j=1}^{n-1}(1-\overline{\lambda_j}\lambda_l)}{1-\overline{\lambda}\lambda_l} \prod_{j=1,j\neq l}^{n} \frac{1}{\lambda_l-\lambda_j}\\ &\quad= \sum_{l=1}^{n-1} \frac{h(\lambda_l)\prod_{j=1}^{n-1}(1-\overline{\lambda_j}\lambda_l)}{1-\overline{\lambda}\lambda_l} \prod_{j=1,j\neq l}^{n} \frac{1}{\lambda_l-\lambda_j} + \frac{h(\lambda_n)\prod_{j=1}^{n-1}(1-\overline{\lambda_j}\lambda_n)}{1-\overline{\lambda}\lambda_n} \prod_{j=1}^{n-1} \frac{1}{\lambda_n-\lambda_j}.\end{aligned}$$

Now

$$\begin{aligned}\prod_{j=1}^{n-1}(1-\overline{\lambda_j}\lambda_l) \prod_{j=1,j\neq l}^{n} \frac{1}{\lambda_l-\lambda_j} &= \frac{1-|\lambda_l|^2}{\lambda_l-\lambda_n} \prod_{j=1,j\neq l}^{n-1} \frac{1-\overline{\lambda_j}\lambda_l}{\lambda_l-\lambda_j} = \frac{1-|\lambda_l|^2}{\lambda_l-\lambda_n} \left(\prod_{j=1,j\neq l}^{n-1} \frac{1-\overline{\lambda_j}\lambda_l}{\lambda_l-\lambda_j}\right) \frac{1-\overline{\lambda_n}\lambda_l}{1-\overline{\lambda_n}\lambda_l}\\ &= \frac{1-|\lambda_l|^2}{1-\overline{\lambda_n}\lambda_l} \frac{1}{(B_n)_{\lambda_l}(\lambda_l)}.\end{aligned}$$

Also,

$$\prod_{j=1}^{n-1}(1-\overline{\lambda_j}\lambda_n) \prod_{j=1}^{n-1} \frac{1}{\lambda_n-\lambda_j} = \frac{1}{(B_n)_{\lambda_n}(\lambda_n)} = \frac{1-|\lambda_n|^2}{1-\overline{\lambda_n}\lambda_n} \frac{1}{(B_n)_{\lambda_n}(\lambda_n)}.$$

Hence

$$\begin{aligned}&\sum_{l=1}^{n} \frac{h(\lambda_l)\prod_{j=1}^{n-1}(1-\overline{\lambda_j}\lambda_l)}{1-\overline{\lambda}\lambda_l} \prod_{j=1,j\neq l}^{n} \frac{1}{\lambda_l-\lambda_j}\\ &\quad= \sum_{l=1}^{n-1} \frac{h(\lambda_l)}{(1-\overline{\lambda_j}\lambda_l)} \frac{1-|\lambda_l|^2}{1-\overline{\lambda_n}\lambda_l} \frac{1}{(B_n)_{\lambda_l}(\lambda_l)} + \frac{h(\lambda_n)}{(1-\overline{\lambda_j}\lambda_n)} \frac{1-|\lambda_n|^2}{1-\overline{\lambda_n}\lambda_n} \frac{1}{(B_n)_{\lambda_n}(\lambda_n)}\end{aligned}$$

which concludes the proof. ∎

Remark 3.6. It is worth reminding the reader again that we are assuming $h \in K_I$ and $A_{\overline{h}}$ is bounded on K_I.

Proof of Theorem 1.7. By Corollary 2.7 the existence of finite non-tangential boundary limits of all functions in $\operatorname{Rng} A_{\overline{h}}$ is equivalent to the interpolation problem of finding $k_\zeta^h \in K_I$ such that

$$(1-\overline{\zeta}z)k_\zeta^h - h \in IH^2, \tag{3.7}$$

where I is now a Blaschke product.

Let us use some ideas from [AC70b]. If there is a function $k_\zeta^h \in K_I$ satisfying (3.7) then there are complex coefficients c_n such that

$$k_\zeta^h = \sum_{n\geq 1} c_n \gamma_n \tag{3.8}$$

with $\sum_{n\geq 1} |c_n|^2 < \infty$. In particular,

$$\overline{c}_n = \langle \gamma_n, k_\zeta^h \rangle.$$

But since $\gamma_n \in K_I$ we can use Corollary 2.9 to get

$$\langle \gamma_n, k_\zeta^h \rangle = (A_{\overline{h}}\gamma_n)(\zeta)$$

which proves the necessity.

Let us now prove the sufficiency.

Assuming $\sum_{n\geq 1} |(A_{\overline{h}}\gamma_n)(\zeta)|^2 < \infty$, we can define the function

$$u = \sum_{n\geq 1} \overline{(A_{\overline{h}}\gamma_n)(\zeta)}\gamma_n \tag{3.9}$$

in K_I. In order to verify the interpolating condition in (3.7), it is sufficient to check that

$$u - \frac{h}{1 - z\overline{\zeta}}$$

vanishes to the right order, meaning that at each point $\lambda \in \Lambda$ these differences vanish with order corresponding to the multiplicity of λ. The reader might observe that these differences are not necessarily in H^2. However, it is clear that $h(z)/(1-\overline{\zeta}z)$ is controlled by $1/(1-\overline{\zeta}z)^{3/2}$ so that we can write the interpolation condition as

$$u - \frac{h}{1 - z\overline{\zeta}} \in IH^p$$

for $p < 2/3$, but we will not really use this formulation.

The proof of the interpolating condition will be very technical in the general case. However, if the zeros are *simple*, which we assume to be the case for the moment, then the formula for $A_{\overline{h}}\gamma_n(\zeta)$ in Corollary 3.5 simplifies the argument considerably. In this situation, we have

$$(A_{\overline{h}}\gamma_n)(\zeta) = \sqrt{1-|\lambda_n|^2}\sum_{l=1}^{n} \frac{\overline{h(\lambda_l)}}{1-\overline{\lambda_l}\zeta}\frac{1}{\overline{(B_n)_{\lambda_l}(\lambda_l)}}\frac{1-|\lambda_l|^2}{1-\overline{\lambda_l}\lambda_n}.$$

Hence using Fubini's theorem we get, for each $N \in \mathbb{N}$,

$$\begin{aligned} u(\lambda_N) &= \sum_{n=1}^{N} \overline{(A_{\overline{h}}\gamma_n)(\zeta)}\gamma_n(\lambda_N) \\ &= \sum_{n=1}^{N} \sqrt{1-|\lambda_n|^2}\sum_{l=1}^{n} \frac{h(\lambda_l)}{1-\lambda_l\overline{\zeta}}\frac{1}{(B_n)_{\lambda_l}(\lambda_l)}\frac{1-|\lambda_l|^2}{1-\overline{\lambda_n}\lambda_l}\frac{\sqrt{1-|\lambda_n|^2}}{1-\overline{\lambda_n}\lambda_N}B_{n-1}(\lambda_N) \\ &= \sum_{l=1}^{N} \frac{h(\lambda_l)}{1-\lambda_l\overline{\zeta}} \underbrace{\sum_{n=l}^{N} \frac{1-|\lambda_l|^2}{1-\overline{\lambda_n}\lambda_l}\frac{1-|\lambda_n|^2}{1-\overline{\lambda_n}\lambda_N}\frac{B_{n-1}(\lambda_N)}{(B_n)_{\lambda_l}(\lambda_l)}}_{=:\,\alpha_{l,N}}. \end{aligned} \tag{3.10}$$

So, in order to show the interpolation condition $u(\lambda_N) = h(\lambda_N)/(1-\overline{\zeta}\lambda_N)$ it suffices to show that

$$\alpha_{l,N} = \begin{cases} 1 & \text{if } l = N \\ 0 & \text{if } l < N. \end{cases}$$

Clearly, if $l = N$ then $\alpha_{N,N} = 1$ (observe in particular that $l = n = N$ and $(B_N)_{\lambda_N} = B_{N-1}$).

Now let $k_{r\xi}(z) = 1/(1-r\overline{\xi}z)$ be the reproducing kernel for H^2 at $r\xi$ for any $\xi \in \mathbb{T}$. Let P_{B_N} be the orthogonal projection onto K_{B_N} which can be written explicitly using the Takenaka-Malmquist-Walsh basis so that

$$v_r := P_{B_N} k_{r\xi} = \sum_{n=1}^{n} \langle k_{r\xi}, \gamma_n \rangle \gamma_n = \sum_{n=1}^{N} \overline{\gamma_n(r\xi)}\gamma_n.$$

Since $v_r - k_{r\xi} \in \ker P_{B_N} = B_N H^2$ we get $v_r(\lambda_n) = k_{r\xi}(\lambda_n)$ for $n = 1, \dots, N$. All functions involved are rational function with no poles on $\mathbb{D}^-$ so that we can pass to the limit as $r \to 1^-$ so that

$$v(\lambda_n) = \lim_{r\to 1^-} v_r(\lambda_n) = \frac{1}{1-\overline{\xi}\lambda_n}, \quad n = 1, 2, \dots, N.$$

Notice also that

$$v = \sum_{n=1}^{N} \overline{\gamma_n(\xi)}\gamma_n = \sum_{n=1}^{N} \overline{(A_{\overline{1}}\gamma_n)(\xi)}\gamma_n.$$

Replacing the function h by 1 in (3.10), we obtain

$$v(\lambda_N) = \sum_{l=1}^{N} \frac{1}{1-\lambda_l\overline{\xi}} \underbrace{\sum_{n=l}^{N} \frac{1-|\lambda_l|^2}{1-\overline{\lambda_n}\lambda_l} \frac{1-|\lambda_n|^2}{1-\overline{\lambda_n}\lambda_N} \frac{B_{n-1}(\lambda_N)}{(B_n)_{\lambda_l}(\lambda_l)}}_{=\alpha_{l,N}},$$

and since $v(\lambda_N) = 1/(1-\overline{\zeta}\lambda_N)$ and $\alpha_{N,N} = 1$, we get

$$\sum_{l=1}^{N-1} \frac{1}{1-\lambda_l\overline{\xi}} \alpha_{l,N} = 0$$

for every ξ. The reproducing kernels for different ξ are linearly independent, so that the coefficients $\alpha_{l,N}$ must necessarily vanish for $l = 1, 2, \dots, N-1$, which finishes the proof for simple zeros.

The reader might observe that the explicit form of $\alpha_{l,N}$ is not really of importance (well, it is, of course...). The central point is that $\alpha_{N,N} = 1$. We will now generalize this argument to the case of arbitrary Blaschke products. As to be expected, the proof is more technical.

For the proof in the general situation, let $\mu = \lambda_{N+1}$ be any point of the sequence such that $\mu \neq \lambda_l$ for every $l \leq N$. It is the first time we meet this zero. Suppose also that μ has multiplicity k_0. We have to show that for every $1 \leq k \leq k_0$,

$$u^{(k-1)}(\mu) = \left(\frac{h}{1-\overline{\zeta}z}\right)^{(k-1)}(\mu).$$

Let us compute the derivatives of u. Let $\prod_{n=1}^{N+k}(z-\lambda_n) = \prod_{l=1}^{r}(z-\mu_l)^{k_l}$ where $\mu_r = \mu$ and $k_r = k$ (and not k_0). Evaluating the $(k-1)$-st derivative of the function u, as defined in (3.9), at μ needs

only to take into account the first $N+k$ terms of the sum since for $n \geq N+k+1$, γ_n has a zero of sufficiently high order at μ that $\gamma_n^{(k-1)}(\mu) = 0$. Thus from (3.4) we get

$$\begin{aligned}
u^{(k-1)}(\mu) &= \sum_{n=1}^{N+k} \overline{(A_{\overline{h}}\gamma_n)(\zeta)}\gamma_n^{(k-1)}(\mu) \\
&= \sum_{n=1}^{N+k}(1-|\lambda_n|^2)\sum_{l=1}^{r}\frac{1}{(k_l-1)!}\frac{d^{k_l-1}}{d\mu_l^{k_l-1}}\left[\frac{h(\mu_l)\prod_{m=1}^{n-1}(1-\overline{\lambda_m}\mu_l)}{(1-\overline{\zeta}\mu_l)\prod_{j=1,j\neq l}^{r}(\mu_l-\mu_j)^{k_j}}\right]\left[k_{\lambda_n}B_{n-1}\right]^{(k-1)}(\mu) \\
&= \sum_{n=1}^{N+k}(1-|\lambda_n|^2)\sum_{l=1}^{r}\frac{1}{(k_l-1)!}\sum_{p=0}^{k_l-1}\binom{k_l-1}{p}\frac{d^p}{d\mu_l^p}\left[\frac{h(\mu_l)}{1-\overline{\zeta}\mu_l}\right]\times \\
&\qquad \times\frac{d^{k_l-1-p}}{d\mu_l^{k_l-1-p}}\left[\frac{\prod_{m=1}^{n-1}(1-\overline{\lambda_m}\mu_l)}{(1-\overline{\lambda}\mu_l)\prod_{j=1,j\neq l}^{r}(\mu_l-\mu_j)^{k_j}}\right]\left[k_{\lambda_n}B_{n-1}\right]^{(k-1)}(\mu).
\end{aligned}$$

We will now apply Fubini's theorem. In order to do this, we observe that the double sum $\sum_{l=1}^{r}\sum_{p=0}^{k_l-1}$ runs exactly through the zeros λ_n, $n = 1,2,...,N+k$. Let us define a function in two variables by

$$\sigma(l,p) = (p+1) + \sum_{j=1}^{l-1} k_l$$

which is a bijection of a disjoint union of sets $\tau_l = \{0,1\ldots,k_l-1\}$, $l = 1,\ldots,r$ to the set $\{1,2,\ldots,N+k\}$. Hence

$$\begin{aligned}
u^{(k-1)}(\mu) &= \sum_{n=1}^{N+k} \overline{(A_{\overline{h}}\gamma_n)(\zeta)}\gamma_n^{(k-1)}(\mu) \\
&= \sum_{l=1}^{r}\frac{1}{(k_l-1)!}\sum_{p=0}^{k_l-1}\binom{k_l-1}{p}\frac{d^p}{d\mu_l^p}\left[\frac{h(\mu_l)}{1-\overline{\zeta}\mu_l}\right]\times \\
&\qquad \times\sum_{n=\sigma(l,p)}^{N+k}(1-|\lambda_n|^2)\frac{d^{k_l-1-p}}{d\mu_l^{k_l-1-p}}\left[\frac{\prod_{m=1}^{n-1}(1-\overline{\lambda_m}\lambda_l)}{\prod_{j=1,j\neq l}^{r}(\mu_l-\mu_j)^{k_j}}\right]\left[k_{\lambda_n}B_{n-1}\right]^{(k-1)}(\mu). \qquad (3.11)
\end{aligned}$$

Let us investigate the term we are particularly interested in for the interpolation problem. It corresponds to the very last term: $l = r$ and $p = k_r - 1 = k-1$. In this situation, $n = \sigma(r,k-1) = N+k$. We compute the last factor:

$$\left[k_{\lambda_n}B_{n-1}\right]^{(k-1)}(\mu) = \sum_{p=0}^{k-1}\binom{k-1}{p}k_{\lambda_{N+k}}^{(p)}B_{N+k-1}^{(k-1-p)}(\mu).$$

Now $B_{N+k-1} = b_\mu^{k-1}\prod_{l=1}^{r-1}b_{\mu_l}^{k_l}$ so that all derivatives up to order $k-2$ of this product evaluated at μ will vanish and

$$B_{N+k-1}^{(k-1)}(\mu) = (b_\mu^{k-1})^{(k-1)}(\mu)\prod_{l=1}^{r-1}b_{\mu_l}^{k_l}(\mu).$$

It is well known, and easy to verify (e.g. using once again the Leibniz rule), that

$$(b_\mu^{k-1})^{(k-1)}(\mu) = \frac{(k-1)!}{(1-|\mu|^2)^{k-1}}.$$

Hence

$$[k_{\lambda_n} B_{n-1}]^{(k-1)}(\mu) = k_\mu(\mu)\frac{(k-1)!}{(1-|\mu|^2)^{k-1}}\prod_{l=1}^{r-1} b_{\mu_l}^{k_l}(\mu) = \frac{(k-1)!}{(1-|\mu|^2)^k}\prod_{l=1}^{r-1} b_{\mu_l}^{k_l}(\mu).$$

We are now in a position to compute the coefficient of the term $\frac{d^{k-1}}{d\mu^{k-1}}\frac{h(\mu)}{1-\overline{\zeta}\mu}$ (corresponding to $l=r$, $k_l=k$, $p=k-1$, and hence, as already seen, $n=\sigma(l,p)=N+k$, $\lambda_{N+k}=\mu$). It is given by

$$\begin{aligned}
&\frac{1}{(k-1)!}\binom{k-1}{k-1}(1-|\mu|^2)\frac{d^{k-1-p}}{d\mu^{k-1-p}}\left[\frac{\prod_{m=1}^{n-1}(1-\overline{\lambda_m}\mu)}{\prod_{j=1,j\neq l}^{r}(\mu-\mu_j)^{k_j}}\right][k_{\lambda_n}B_{n-1}]^{(k-1)}(\mu)\\
&\quad = \frac{1}{(k-1)!}(1-|\mu|^2)\left[\frac{\prod_{m=1}^{r-1}(1-\overline{\mu_m}\mu)^{k_m}(1-|\mu|^2)^{k-1}}{\prod_{j=1}^{r-1}(\mu-\mu_j)^{k_j}}\right]\frac{(k-1)!}{(1-|\mu|^2)^k}\prod_{l=1}^{r-1} b_{\mu_l}^{k_l}(\mu)\\
&\quad = 1. \qquad (3.12)
\end{aligned}$$

Hence we are led to show that the remaining sum adds up to 0. For this, it is sufficient to show that for every $l=1,\dots,r-1$, $p=0,\dots,k_l-1$ and for $l=r$, $p=0,\dots,k-2$, we have

$$\sum_{n=\sigma(l,p)}^{N+k}(1-|\lambda_n|^2)\frac{d^{k_l-1-p}}{d\mu_l^{k_l-1-p}}\left[\frac{\prod_{m=1}^{n-1}(1-\overline{\lambda_m}\lambda_l)}{\prod_{j=1,j\neq l}^{r}(\mu_l-\mu_j)^{k_j}}\right][k_{\lambda_n}B_{n-1}]^{(k-1)}(\mu)=0. \qquad (3.13)$$

The trick is the same as for simple zeros: redo all the computations from above for the case $h=1$ for which we will deduce (3.13) from the interpolating property.

For this, let $\xi\in\mathbb{T}$ and $0<r<1$. Set

$$v_r(z)=\sum_{n=1}^{N+k}\overline{(A_{\overline{1}}\gamma_n)(r\xi)}\gamma_n(z)=\sum_{n=1}^{N+k}\overline{\gamma_n(r\xi)}\gamma_n(z).$$

Let us first check that v_r interpolates what it should (while this is of course contained in [AC70b] we add here a proof for completeness). To this end, as before, let $k_{r\xi}$ be the H^2 reproducing kernel at $r\xi\in\mathbb{D}$. Also let $P_{B_{N+k}}$ be the orthogonal projection onto the space $K_{B_{N+k}}$. Using the Takenaka-Malmquist-Walsh functions we obtain

$$P_{B_{N+k}}k_{r\xi}=\sum_{n=0}^{N+k}\langle k_{r\xi},\gamma_n\rangle\gamma_n=\sum_{n=0}^{N+k}\overline{\gamma_n(r\xi)}\gamma_n=v_r.$$

Hence

$$v_r-k_{r\xi}\in\ker P_{B_{N+k}}=B_{N+k}H^2.$$

Now all these functions are rational functions with no poles in $\mathbb{D}$ so that we can pass to the limit $r\to 1^-$ to obtain for μ and $1\le k\le k_0$ (same meaning of these parameters as in the first part of the proof),

$$v^{(k-1)}(\mu)=k_\xi^{(k-1)}(\mu)=\frac{d^{k-1}}{d\mu^{k-1}}k_\xi(\mu).$$

(Note that again the difference $v-k_\xi$ is not in H^2 since k_ξ is not.)

Exactly as in (3.11) we obtain

$$\begin{aligned} v^{(k-1)}(\mu) &= \sum_{n=1}^{N+k} \overline{(A_{\overline{1}}\gamma_n)(\xi)}\gamma_n^{(k-1)}(\mu) \\ &= \sum_{l=1}^{r} \frac{1}{(k_l-1)!} \sum_{p=0}^{k_l-1} \binom{k_l-1}{p} \frac{d^p}{d\mu_l^p}\left[\frac{1}{1-\overline{\xi}\mu_l}\right] \times \\ &\qquad \times \sum_{n=\sigma(l,p)}^{N+k} (1-|\lambda_n|^2) \frac{d^{k_l-1-p}}{d\mu_l^{k_l-1-p}}\left[\frac{\prod_{m=1}^{n-1}(1-\overline{\lambda_m}\lambda_l)}{\prod_{j=1,j\neq l}^{r}(\mu_l-\mu_j)^{k_j}}\right] \left[k_{\lambda_n} B_{n-1}\right]^{(k-1)}(\mu). \end{aligned}$$

The leading coefficient for $l = r$ and $p = k_r - 1 = k - 1$ has already been computed in (3.12) to be 1. Hence subtracting the term corresponding to the leading coefficient, we obtain (splitting the sum into the terms for $l \in \{1, 2, \dots, r-1\}$ and $l = r$)

$$\begin{aligned} 0 &= \sum_{l=1}^{r-1} \frac{1}{(k_l-1)!} \sum_{p=0}^{k_l-1} \binom{k_l-1}{p} \frac{d^p}{d\mu_l^p}\left[\frac{1}{1-\overline{\xi}\mu_l}\right] \times \\ &\qquad \times \sum_{n=\sigma(l,p)}^{N+k} (1-|\lambda_n|^2) \frac{d^{k_l-1-p}}{d\mu_l^{k_l-1-p}}\left[\frac{\prod_{m=1}^{n-1}(1-\overline{\lambda_m}\mu_l)}{\prod_{j=1,j\neq l}^{r}(\mu_l-\mu_j)^{k_j}}\right] \left[k_{\lambda_n} B_{n-1}\right]^{(k-1)}(\mu) \\ &+ \frac{1}{(k-1)!} \sum_{p=0}^{k-2} \binom{k-1}{p} \frac{d^p}{d\mu^p}\left[\frac{1}{1-\overline{\xi}\mu}\right] \times \\ &\qquad \times \sum_{n=\sigma(l,p)}^{N+k} (1-|\lambda_n|^2) \frac{d^{k_l-1-p}}{d\mu_l^{k_l-1-p}}\left[\frac{\prod_{m=1}^{n-1}(1-\overline{\lambda_m}\mu_l)}{\prod_{j=1}^{r-1}(\mu-\mu_j)^{k_j}}\right] \left[k_{\lambda_n} B_{n-1}\right]^{(k-1)}(\mu). \end{aligned}$$

The above formula is valid for every $\xi \in \mathbb{T}$. Now, observe that the functions

$$\xi \longmapsto \frac{d^p}{d\mu_l^p}\left[\frac{1}{1-\overline{\xi}\mu_l}\right]$$

form a linearly independant family for $l = 1, ..., r-1$, $p = 0, ..., k_l - 1$ and for $l = r$, $p = 0, ..., k-2$, implying that the coefficients

$$\sum_{n=\sigma(l,p)}^{N+k} \frac{d^{k_l-1-p}}{d\mu_l^{k_l-1-p}}\left[\frac{\prod_{m=1}^{n-1}(1-\overline{\lambda_m}\mu_l)}{\prod_{j=1}^{r-1}(\mu-\mu_j)^{k_j}}\right] \left[k_{\lambda_n} B_{n-1}\right]^{(k-1)}(\mu)$$

have to vanish in the required ranges of the parameters of l, p. ∎

Remark 3.14. From the identity $k_\lambda^h = P_I(hk_\lambda^I)$ from (2.5), and the fact that $\{\gamma_n : n \in \mathbb{N}\}$ forms an orthonormal basis for K_I, we see that

$$\begin{aligned} \|k_\lambda^h\|^2 &= \|P_I(hk_\lambda^I)\|^2 = \sum_{n\geq 1} |\langle P_I(hk_\lambda^I), \gamma_n\rangle|^2 = \sum_{\geq 1} |\langle k_\lambda^I, \overline{h}\gamma_n\rangle|^2 \\ &= \sum_{\geq 1} |(A_{\overline{h}}\gamma_n)(\lambda)|^2. \end{aligned}$$

From (3.8) it follows that

$$\|k_\zeta^h\|^2 = \sum_{n\geq 1} |(A_{\overline{h}}\gamma_n)(\zeta)|^2.$$

Proof of Theorem 1.9. Instead of deriving this result from our main theorem, the idea is to use directly the interpolation property as in [AC70b]. The existence of the boundary limits is as before equivalent to the existence of the function k_ζ^h. And from the interpolation condition (3.7) the existence of the function k_ζ^h is equivalent to the solution of the problem

$$k_\zeta^h(\lambda_n) = \frac{h(\lambda_n)}{1-\overline{\zeta}\lambda_n}, \quad n \geq 1.$$

From Shapiro-Shields [SS61], this is equivalent to

$$\sum_{n\geq 1}(1-|\lambda|^2)\left|\frac{h(\lambda_n)}{\zeta-\lambda_n}\right|^2 < \infty.$$

This proves the result. ∎

4. Some examples

Recall that a truncated Toeplitz operators with co-analytic symbol $\overline{h}$ is just the restriction of the regular Toeplitz operator with symbol $\overline{h}$ to a model space K_I. Let us discuss some simple examples which illustrate the smoothing effects of applying Toeplitz operators to functions in K_I or even H^2.

Example 4.1. The following general fact is well known for functions f in H^2 :

$$|f(z)| = O\left(\frac{1}{\sqrt{1-|z|}}\right). \tag{4.2}$$

It can actually be shown that this growth condition can be replaced in a non-tangential approach region by a little-oh condition:

$$|f(\lambda)| = o\left(\frac{1}{\sqrt{1-|\lambda|}}\right), \quad \lambda \xrightarrow{\angle} \zeta \in \mathbb{T}. \tag{4.3}$$

The notation $z \xrightarrow{\angle} \zeta$ means that z tends non tangentially to ζ. As a consequence, we observe that if φ is anaytic and $|\varphi(z)| \leq \sqrt{1-|z|}$ as $z \xrightarrow{\angle} \zeta$ — which is for instance the case when $\varphi(z) = \sqrt{\zeta - z}$ — then every function in the range of the *analytic* Toeplitz operator $T_\varphi f$, where actually $f \in H^2$, will have a boundary limit (zero) at ζ.

Example 4.4. The situation is more intricate when considering co-analytic symbols. A simple observation is the following: If $h(z) = 1 - z$, then for every function $f \in H^2$,

$$T_{\overline{h}}f(z) = T_{\overline{1-z}}f(z) = f(z) - \frac{f(z)-f(0)}{z} = \frac{f(z)(z-1)}{z} - \frac{f(0)}{z}$$

which tends in fact to $-f(0)$ (and which is in general not 0) when $z \xrightarrow{\angle} 1$ (we have used (4.2)).

Example 4.5. In this example we use Theorem 1.9 to show that the natural multiplier $\sqrt{1-z}$, which makes every function in H^2 vanish non-tangentially at 1 (as observed in Example 4.1), is not sufficient for co-analytic Toeplitz operators.

Let $\Lambda = (1-\frac{1}{2^n})_{n\geq 1}$, I be the Blaschke product with these zeros, and $h_\varepsilon(z) = (1-z)^{1/2+\varepsilon}$. Then every function $f \in \operatorname{Rng} A^2_{\overline{h_\varepsilon}}$ has non-tangential limit in 1, if and only if the condition in (1.10) is fulfilled. Now observe that

$$\sum_{n\geq 1}(1-|\lambda_n|^2)\left|\frac{h_\varepsilon(\lambda_n)}{1-\lambda_n}\right|^2 \simeq \sum_{n\geq 1} 2^{-n}\left|\frac{1/2^{n(1/2+\varepsilon)}}{1/2^n}\right|^2 = \sum_{n\geq 1} 2^{-2n\varepsilon}$$

which converges if and only if $\varepsilon > 0$. So we need the symbol h to decrease faster than $\sqrt{1-z}$ to ensure existence of the boundary limits of functions in $\operatorname{Rng} A^2_{\overline{h}}$.

It is possible to consider a decrease closer to $\sqrt{1-z}$, for example

$$h(z) = \log 2\frac{\sqrt{1-z}}{(-\log(1-z))}$$

(for which $h(\lambda_n) = 1/(n2^{n/2})$), but we can never reach $\sqrt{1-z}$.

Example 4.6. In this next proposition, we see that Theorem 1.9 is not true for non-interpolating Blaschke sequences.

Proposition 4.7. *There exists a point $\zeta \in \mathbb{T}$, a Blaschke product I whose zeros $\Lambda \subset \mathbb{D}$ satisfy the condition* (1.10) *at ζ, a function $h \in K_I$ such that $A^2_{\overline{h}}$ is bounded on K_I. Still there are functions in $A^2_{\overline{h}}K_I$ which do not have finite non-tangential boundary limits at ζ.*

Proof. The proof of this result relies on a result concerning interpolation on finite unions of interpolating sequences [BNØ96, Har96]. Let $\Lambda_1 = \{\lambda^1_n\}_{n\geq 1} = \{1-1/2^n\}_{n\geq 1}$, which is an interpolating sequence [Gar07] and let $\Lambda_2 = \{\lambda^2_n\}_{n\geq 1}$ satisfy $|b_{\lambda^1_n}(\lambda^2_n)| = 1/n$. The sequence Λ_2 is a sufficiently small perturbation of Λ_1 such that Λ_2 will also be interpolating. Also note that the sequence $\Lambda := \Lambda_1 \cup \Lambda_2$ accumulates non tangentially at $\zeta = 1$. Let

$$v^1_n = \frac{1}{n2^{n/2}}\,, \quad v^2_n = 0.$$

The central result used here is the following: a sequence of values $(w^k_n)_{n\geq 1;k=1,2}$ is a trace of a function $f \in H^2$ (or K_I) if and only if [BNØ96, Har96]

$$\sum_{n\geq 1}(1-|\lambda^1_n|^2)\left[|w^1_n|^2 + \left|\frac{w^2_n - w^1_n}{b_{\lambda^1_n}(\lambda^2_n)}\right|^2\right] < \infty. \tag{4.8}$$

For the values $w^i_n = v^i_n$ that we have given above, we get:

$$\sum_{n\geq 1}(1-|\lambda^1_n|^2)\left[|v^1_n|^2 + \left|\frac{v^2_n - v^1_n}{b_{\lambda^1_n}(\lambda^2_n)}\right|^2\right] \simeq \sum_{n\geq 1}\frac{1}{2^n}\left[\left(\frac{1}{n2^{n/2}}\right)^2 + \left|\frac{1/(n2^{n/2})}{1/n}\right|^2\right] < \infty,$$

and so that there is a function h in H^2 or in K_I taking the values v^i_n at λ^i_n.

Next we check the condition (1.10). Note that since $h(\lambda^2_n) = v^2_n = 0$, we only have to sum over Λ_1. Indeed we get

$$\sum_{n\geq 1}(1-|\lambda^1_n|^2)\left|\frac{h(\lambda^1_n)}{1-\lambda^1_n}\right|^2 \simeq \sum_{n\geq 1}\frac{1}{2^n}\left|\frac{1/(n2^{n/2})}{1/2^n}\right|^2 = \sum_{n\geq 1}\frac{1}{n^2} < \infty.$$

Let us check that the sequence defined by

$$w_n^i := \frac{h(v_n^i)}{\zeta - \lambda_n^i}, \quad n \geq 1, i = 1, 2,$$

cannot be realized by a function in K_I so that k_ζ^h does not exist and hence there are functions in $A_{\overline{h}}K_I$ that do not admit boundary limits in $\zeta = 1$. In order to do so, we have to check that this sequence does not satisfy the condition (4.8). We compute to get

$$\begin{aligned}\sum_{n\geq 1}(1-|\lambda_n^1|^2)\left[|w_n^1|^2 + \left|\frac{w_n^2 - w_n^1}{b_{\lambda_n^1}(\lambda_n^2)}\right|^2\right] &\simeq \sum_{n\geq 1}\frac{1}{2^n}\left(\left|\frac{1/(n2^{n/2})}{1/2^n}\right|^2 + \left|\frac{\frac{1/(n2^{n/2})}{1/2^n} - 0}{1/n}\right|^2\right)\\ &= \sum_{n\geq 1}(\frac{1}{n^2}+1) = +\infty\end{aligned}$$

so that this value sequence cannot be realized by a function in K_I.

We finally have to check that $A_{\overline{h}}$ is bounded on K_I. For this, note that K_I is an l^2-sum of K_{B_n} where B_n is the finite Blaschke product with zeros $\{\lambda_n^1, \lambda_n^2\}$ (see [Nik02, Theorem C3.2.14]). By this we mean that every $f \in K_I$ can be written as

$$f = \sum_{n\geq 1} f_n, \quad f_n \in K_{B_n}, \quad \|f\|^2 \asymp \sum_{n\geq 1}\|f_n\|^2.$$

We use the Takenaka-Malmquist-Walsh system to generate K_{B_n}:

$$\gamma_{n,1}(z) = \frac{\sqrt{1-|\lambda_n^1|^2}}{1-\overline{\lambda_n^1}z}, \quad \gamma_{n,2}(z) = \frac{\sqrt{1-|\lambda_n^2|^2}}{1-\overline{\lambda_n^2}z}\frac{z-\lambda_n^1}{1-\overline{\lambda_n^1}z}.$$

So, every function $f \in K_I$ can be written as

$$f = \sum_{n\geq 1}(\alpha_{n,1}\gamma_{n,1} + \alpha_{n,2}\gamma_{n,2})$$

with $\|f\|^2 \simeq \sum_{n\geq 1}|\alpha_{n,1}|^2 + |\alpha_{n,2}|^2 < \infty$. Apply now $A_{\overline{h}}$ to this sum (we could start with finite sums and check that we have a uniform norm control). Clearly

$$A_{\overline{h}}\gamma_{n,1} = \overline{h(\lambda_n^1)}\gamma_{n,1}.$$

The action of $A_{\overline{h}}\gamma_{n,2}$ can be deduced from Corollary 3.5. We obtain

$$(A_{\overline{h}}\gamma_{n,2})(z) = \sqrt{1-|\lambda_n^2|^2}\left[\frac{\overline{h(\lambda_n^1)}}{1-\overline{\lambda_n^1}z}\frac{1}{\overline{b_{\lambda_n^2}(\lambda_n^1)}}\frac{(1-|\lambda_n^1|^2)}{1-\overline{\lambda_n^1}\lambda_n^2} + \frac{\overline{h(\lambda_n^2)}}{1-\overline{\lambda_n^1}z}\frac{1}{\overline{b_{\lambda_n^1}(\lambda_n^2)}}\right].$$

Note that $h(\lambda_n^2) = v_n^2 = 0$, and hence

$$(A_{\overline{h}}\gamma_{n,2})(z) = \sqrt{1-|\lambda_n^2|^2}\frac{\overline{h(\lambda_n^1)}}{1-\overline{\lambda_n^1}z}\frac{1}{\overline{b_{\lambda_n^2}(\lambda_n^1)}}\frac{(1-|\lambda_n^1|^2)}{1-\overline{\lambda_n^1}\lambda_n^2} = \beta_n\gamma_{n,1},$$

where

$$\beta_n = \frac{\sqrt{1-|\lambda_n^2|^2}\sqrt{1-|\lambda_n^1|^2}}{1-\overline{\lambda_n^1}\lambda_n^2}\frac{\overline{h(\lambda_n^1)}}{\overline{b_{\lambda_n^2}(\lambda_n^1)}}.$$

In view of the explicit values of λ_n^i, $h(\lambda_n^i)$ and $|b_{\lambda_n^2}(\lambda_n^1)|$, the sequence $\{\beta_n\}_{n\geq 1}$ is bounded (it actually tends to zero quickly). We thus get

$$A_{\overline{h}}^* f = \sum_{n\geq 1} \left(\alpha_{n,1}\overline{h(\lambda_n^1)}\gamma_{n,1} + \alpha_{n,2}\beta_n\gamma_{n,1}\right) = \sum_{n\geq 1} \left(\alpha_{n,1}\overline{h(\lambda_n^1)} + \alpha_{n,2}\beta_n\right)\gamma_{n,1}$$

and hence, since h is also bounded on Λ_1 (actually decreasing very fast to 0),

$$\|A_{\overline{h}}^* f\|^2 = \sum_{n\geq 1} |\overline{h(\lambda_n^1)}\alpha_{n,1} + \beta_n\alpha_{n,2}|^2 \lesssim \sum_{n\geq 1} |\alpha_{n,1}|^2 + |\alpha_{n,2}|^2 \simeq \|f\|^2.$$

■

Note how in this example B does not have non-tangential limit at $\zeta = 1$. Indeed, B vanishes at its zeros and, in the middle between two successive pairs $\{\lambda_n^1, \lambda_n^2\}$ and $\{\lambda_{n+1}^1, \lambda_{n+1}^2\}$, we are far from the elements of the two interpolating sequences Λ_1 and Λ_2. Thus B will be big at these points.

The second remark is that for $h(z) = 1-z$ we have already seen that every function in $\operatorname{Rng} A_{\overline{1-z}}$ will have a limit at $\zeta = 1$. Choosing, as mentioned in Example 4.5, $h(z) = \log 2\sqrt{1-z}/(-\log(1-z))$ (which gives exactly $h(\lambda_n^1) = 1/(n2^{n/2})$) and $w_n^i = h(\lambda_n^i)$, it can be checked that (4.8) is true so that for this function h, every $f \in A_{\overline{h}}^* K_I$ has non tangential limit at $\zeta = 1$. If the reader prefers a function in K_I, it is sufficient to project h into K_I which does not change the values on Λ.

The arguments given in the proof of Proposition 4.7 indicate how to adapt the construction to generalize Theorem 1.9 to finite unions of interpolating sequences.

Example 4.9. In this final example, we apply Theorem 1.7 to a sequence which is not a finite union of interpolating sequences. Fix $\beta \in (1/2, 1)$. Let

$$\lambda_n = 1 - \frac{1}{2^{n^\beta}},$$

and let B be the Blaschke product associated with the sequence $\Lambda = \{\lambda_n\}_n$. Since the convergence of this sequence to 1 is sub-exponential, there will be dyadic intervals $[1-1/2^n, 1-1/2^{n+1}]$ in the radius $[0,1)$ containing arbitrarily big numbers of elements of Λ so that the associated measure $\sum_{n\geq 1}(1-|\lambda_n|^2)\delta_{\lambda_n}$ cannot be Carleson (see [Gar07] for more information on Carleson measures).

Let us first estimate $|B_{\lambda_n}(\lambda_n)|$ where B_{λ_n} is the Blaschke product associated with the sequence $\Lambda \setminus \{\lambda_n\}$. In order to do these estimates, we will consider

$$\begin{aligned}\log|B_{\lambda_n}(\lambda_n)|^{-1} &= \sum_{k\neq n} \log|b_{\lambda_k}(\lambda_n)|^{-1} = \sum_{k\neq n} \log\left|\frac{1/2^{n^\beta} - 1/2^{k^\beta}}{1/2^{n^\beta} + 1/2^{k^\beta} - 1/2^{k^\beta+n^\beta}}\right|^{-1} \\ &= \sum_{k\neq n} \log\left|\frac{2^{n^\beta} + 2^{k^\beta} - 1}{2^{n^\beta} - 2^{k^\beta}}\right|.\end{aligned}$$

We can suppose that n is large enough so that we do not have to worry about the -1 which occurs in the last numerator. We will now split the summation (in the index k) into 4 (or 2) pieces.

Case 1: Consider $n+1 \leq k \leq n + n^{1-\beta}$. Then

$$\left|\frac{2^{n^\beta} + 2^{k^\beta}}{2^{n^\beta} - 2^{k^\beta}}\right| \simeq \left|\frac{2^{k^\beta}}{2^{n^\beta} - 2^{k^\beta}}\right| = \frac{2^{k^\beta}}{2^{k^\beta} - 2^{n^\beta}} = \frac{1}{1 - 2^{n^\beta - k^\beta}}.$$

Note that

$$\begin{aligned} 0 &\geq n^\beta - k^\beta \geq n^\beta - (n+n^{1-\beta})^\beta = n^\beta - n^\beta(1+n^{-\beta})^\beta \\ &= n^\beta - n^\beta(1+\beta/n^\beta + o(1/n^\beta)) \\ &= -\beta + o(1). \end{aligned} \tag{4.10}$$

So, $-1 < -\ln 2 < -\beta \ln 2 \lesssim (\ln 2)(n^\beta - k^\beta) \leq 0$ (where the "$\lesssim$" is asymptotically, for $n \to \infty$, a "$\leq$"), and hence

$$2^{n^\beta - k^\beta} = e^{(\ln 2)(n^\beta - k^\beta)} \simeq 1 + (\ln 2)(n^\beta - k^\beta),$$

so that

$$\left| \frac{2^{n^\beta} + 2^{k^\beta}}{2^{n^\beta} - 2^{k^\beta}} \right| \simeq \frac{1}{1 - 2^{n^\beta - k^\beta}} \simeq \frac{1}{\ln 2 (k^\beta - n^\beta)}$$

Now, setting $k = n + l$ with $l \in \{1, 2, \ldots, n^{1-\varepsilon}\}$ we get

$$(n+l)^\beta - n^\beta = n^\beta (1 + \frac{l}{n})^\beta - n^\beta \simeq \frac{\beta l}{n^{1-\beta}}.$$

Hence

$$\sum_{k=n+1}^{n+n^{1-\beta}} \log |b_{\lambda_k}(\lambda_n)|^{-1} \simeq \sum_{l=1}^{n^{1-\beta}} \log \frac{n^{1-\beta}}{\beta l \ln 2}.$$

And switching back to the product we get

$$\prod_{k=n+1}^{n+n^{1-\beta}} |b_{\lambda_k}(\lambda_n)|^{-1} \simeq \left(\frac{n^{1-\beta}}{\beta \ln 2} \right)^{n^{1-\beta}} \frac{1}{(n^{1-\beta})!}.$$

Using Stirling's formula

$$\frac{N^N}{N!} \simeq \frac{e^N}{\sqrt{2\pi N}},$$

we obtain with $N = n^{1-\beta}$,

$$\prod_{k=n+1}^{n+n^{1-\beta}} |b_{\lambda_k}(\lambda_n)|^{-1} \simeq \left(\frac{e}{\beta \ln 2} \right)^{n^{1-\beta}} \frac{1}{\sqrt{2\pi n^{1-\beta}}} \lesssim e^{c n^{1-\beta}} \tag{4.11}$$

for some suitable constant c.

Case 2: Suppose now that $k \geq n + n^{1-\beta}$. Observe

$$\left| \frac{2^{n^\beta} + 2^{k^\beta}}{2^{n^\beta} - 2^{k^\beta}} \right| = \left| 1 + 2 \frac{2^{n^\beta}}{2^{k^\beta} - 2^{n^\beta}} \right|.$$

Then

$$\frac{2^{n^\beta}}{2^{k^\beta} - 2^{n^\beta}} \leq \frac{2^{n^\beta}}{2^{(n+n^{1-\beta})^\beta} - 2^{n^\beta}} = \frac{1}{2^{(n+n^{1-\beta})^\beta - n^\beta} - 1},$$

which, by similar computations as in (4.10), is controlled by

$$\frac{1}{2^\beta - 1}.$$

This enables us now to write

$$\log \left| 1 + 2 \frac{2^{n^\beta}}{2^{k^\beta} - 2^{n^\beta}} \right| \simeq \frac{2}{2^{k^\beta - n^\beta} - 1} \lesssim \frac{2^{n^\beta}}{2^{k^\beta}}.$$

Using the estimate

$$\int_M^\infty e^{-x^\beta} dx \simeq M^{1-\beta} e^{-M^\beta},$$

we can compute

$$\sum_{k \geq n+n^{1-\beta}} \log \left| 1 + 2\frac{2^{n^\beta}}{2^{k^\beta} - 2^{n^\beta}} \right| \lesssim 2^{n^\beta} \sum_{k \geq n+n^{1-\beta}} \frac{1}{2^{k^\beta}} \simeq 2^{n^\beta} (n + n^{1-\beta})^{1-\beta} \frac{1}{2^{(n+n^{1-\beta})^\beta}} \simeq n^{1-\beta}$$

so that we also get

$$\prod_{k \geq n+n^{1-\beta}} |b_{\lambda_k}(\lambda_n)|^{-1} \leq e^{cn^{1-\beta}}$$

for some suitable constant c.

We will also include a brief discussion of the cases 3 — $(n - n^{1-\varepsilon}) \leq k \leq n-1$ — and 4 — $1 \leq k \leq (n - n^{1-\varepsilon})$ — which are treated in essentially the same way.

Case 3: Consider $n - n^{1-\beta} \leq k \leq n-1$. Then

$$\left| \frac{2^{n^\beta} + 2^{k^\beta}}{2^{n^\beta} - 2^{k^\beta}} \right| \simeq \left| \frac{2^{n^\beta}}{2^{n^\beta} - 2^{k^\beta}} \right| = \frac{1}{1 - 2^{k^\beta - n^\beta}}.$$

Now

$$0 \geq k^\beta - n^\beta \geq (n - n^{1-\beta})^\beta - n^\beta \simeq -\beta + o(1)$$

as in (4.10). So, $-1 < -\ln 2 < -\beta \ln 2 \lesssim (\ln 2)(k^\beta - n^\beta) \leq 0$ (where the "$\lesssim$" is asymptotically, for $n \to \infty$, a "$\leq$"), and we can conclude as in the case 1 to obtain

$$\prod_{k=n-n^{1-\beta}}^{n-1} |b_{\lambda_k}(\lambda_n)|^{-1} \simeq \left(\frac{e}{\beta \ln 2} \right)^{n^{1-\beta}} \frac{1}{\sqrt{2\pi n^{1-\beta}}} \lesssim e^{cn^{1-\beta}}$$

for some suitable constant c.

Case 4: Suppose now that $k \leq n - n^{1-\beta}$. Observe

$$\left| \frac{2^{n^\beta} + 2^{k^\beta}}{2^{n^\beta} - 2^{k^\beta}} \right| = \left| 1 + 2\frac{2^{k^\beta}}{2^{n^\beta} - 2^{k^\beta}} \right|.$$

Then

$$\frac{2^{k^\beta}}{2^{n^\beta} - 2^{k^\beta}} = \frac{1}{2^{n^\beta - k^\beta} - 1} \leq \frac{1}{2^{n^\beta - (n-n^{1-\beta})^\beta} - 1}$$

which, by similar computations as in (4.10), is controlled by

$$\frac{1}{2^\beta - 1}.$$

This enables us now to write

$$\log \left| 1 + 2\frac{2^{k^\beta}}{2^{n^\beta} - 2^{k^\beta}} \right| \simeq 2\frac{2^{k^\beta}}{2^{n^\beta} - 2^{k^\beta}} \lesssim \frac{2^{k^\beta}}{2^{n^\beta}}.$$

Using

$$\int_1^M e^{x^\beta} dx \simeq M^{1-\beta} e^{M^\beta}$$

we can estimate

$$\sum_{k\le n-n^{1-\beta}} \log\left|1+2\frac{2^{n^\beta}}{2^{k^\beta}-2^{n^\beta}}\right| \lesssim \frac{1}{2^{n^\beta}} \sum_{k\le n-n^{1-\beta}} 2^{k^\beta} \simeq \frac{1}{2^{n^\beta}}(n-n^{1-\beta})^{1-\beta}2^{(n-n^{1-\beta})^\beta} \simeq n^{1-\beta}$$

so that we also get

$$\prod_{k\le n-n^{1-\beta}} |b_{\lambda_k}(\lambda_n)|^{-1} \le e^{cn^{1-\beta}}$$

for some suitable constant c.

Putting this all together we obtain

$$\delta_n := |B_{\lambda_n}(\lambda_n)| \ge e^{-cn^{1-\beta}}$$

for some suitable constant c.

Let us now return to our problem of estimating $|A_{\overline{h}}\gamma_n(\zeta)|$. From Proposition 3.5, we have

$$(A_{\overline{h}}\gamma_n)(\zeta) = \sqrt{1-|\lambda_n|^2}\sum_{l=1}^{n} \frac{\overline{h(\lambda_l)}}{1-\overline{\lambda_l}\zeta}\frac{1}{\overline{(B_n)_{\lambda_l}(\lambda_l)}}\frac{1-|\lambda_l|^2}{1-\overline{\lambda_l}\lambda_n}$$

Our Blaschke product constructed above accumulates at $\zeta = 1$ and contains only points in $(0,1)$. Let $h(z) = (1-z)^{1-\varepsilon}$. Then

$$|(A_{\overline{h}}\gamma_n)(\zeta)| \lesssim \sqrt{1-\lambda_n}\sum_{l=1}^{n}\frac{(1-\lambda_l)^{1-\varepsilon}}{1-\lambda_l}\frac{1}{|(B_n)\lambda_l(\lambda_l)|}\frac{1-\lambda_l}{1-\lambda_l} \lesssim \sqrt{1-\lambda_n}\sum_{l=1}^{n}\frac{(1-\lambda_l)^{-\varepsilon}}{\delta_l}$$

Recall that $\lambda_n = 1-\frac{1}{2^{n^\beta}}$ and $\delta_n \ge e^{-cn^{1-\beta}}$. Hence

$$|(A_{\overline{h}}\gamma_n)(\zeta)| \lesssim \frac{1}{2^{n^\beta/2}}\sum_{l=1}^{n}\frac{2^{\varepsilon l^\beta}}{e^{-cl^{1-\beta}}} \lesssim \frac{1}{2^{n^\beta/2}} n 2^{\varepsilon n^\beta} e^{cn^{1-\beta}} = n2^{\varepsilon n^\beta + c\ln 2 n^{1-\beta} - n^\beta/2}$$

which is square summable as long as $\varepsilon < 1/2$ and $\beta > 1/2$.

Note that again our zeros are contained in the radius $(0,1)$ and the function h has to go slightly faster to zero than the square root as in the situation when Λ was an interpolating Blaschke sequence in $(0,1)$.

Note also that in this example

$$\sphericalangle \lim_{\lambda\to 1} B(\lambda) = 0.$$

5. Unbounded operators

For any $h \in H^2$ the truncated Toeplitz operator $A_{\overline{h}}$ turns out to be a closed, densely defined operator on K_I with a domain $\mathcal{D}(A_{\overline{h}})$ which contains $K_I \cap H^\infty$ [Sar08]. If one looks closely at the proof of the two main theorems of this paper (Theorem 1.7 and Theorem 1.9), one realizes that the sufficiency parts still hold but with $\operatorname{Rng} A_{\overline{h}}$ defined as $A_{\overline{h}}\mathcal{D}(A_{\overline{h}})$.

Furthermore, the conditions given in these theorems are still sufficient for every $h \in H^2$ when $\operatorname{Rng} A_{\overline{h}}$, as defined in the previous paragraph, is replaced by the linear manifold $\{f_h : f \in K_I\}$, where f_h is defined by the left-hand side of (1.5), i.e.,

$$f_h(\lambda) := \langle f, (\mathrm{Id} - \overline{\lambda}A_z)^{-1}P_I h\rangle.$$

Repeating the argument in (1.4), we can also write

$$f_h(\lambda) = \langle f, k_\lambda P_I h\rangle.$$

Note that the linear manifold $\{f_h : f \in K_I\}$ is not necessarily a subset of K_I. However,

$$f_h(\lambda) = \int_{\mathbb{T}} \frac{f(\xi)\overline{(P_I h)(\xi)}}{1-\overline{\xi}\lambda} dm(\xi)$$

is a Cauchy transform of the finite measure $f\overline{P_I h}dm$. Since Cauchy transforms of finite measures on the circle are known to belong to all the Hardy classes H^p for $0 < p < 1$ [CMR06, p. 43], we know that the non-tangential limits of f_h exist almost everywhere. Theorems 1.7 and 1.9 give sufficient conditions when these non-tangential limits exist at specific points of the circle.

6. Open questions

Conspicuously missing from this paper is a discussion of what happens to $\operatorname{Rng} A_{\overline{h}}$ when I is a general inner function $I = Bs_\mu$ and not necessarily a Blaschke product as was discussed here. In this case, if we are aiming for a similar characterization as in Theorems 1.7 and 1.9, we would need a different orthonormal basis than $\{\gamma_n : n \in \mathbb{N}\}$. So suppose that $\{\varphi_n : n \in \mathbb{N}\}$ is an orthonormal basis for K_I for a general inner function I. Some examples can be found in [AC70a]. Then Proposition 2.6 still holds and so the non-tangential boundary values at a fixed point $\zeta \in \mathbb{T}$ will exist for all functions from $\operatorname{Rng} A_{\overline{h}}$ if and only if the kernel functions k_λ^h remain bounded whenever $\lambda \to \zeta$ non-tangentially. The exact same computation as in Remark 3.14 will show that

$$\|k_\lambda^h\|^2 = \sum_{n\geq 1} |(A_{\overline{h}}\varphi_n)(\lambda)|^2.$$

At this point, two problems stand in our way. The first is to prove that $(A_{\overline{h}}\varphi_n)(\zeta)$ exists as it did so nicely for $(A_{\overline{h}}\gamma_n)(\zeta)$. Recall that $A_{\overline{h}}\gamma_n$ is a rational function whose poles are off of $\mathbb{D}^-$. Is $A_{\overline{h}}\varphi_n$ such a nice function so we can compute $(A_{\overline{h}}\varphi_n)(\zeta)$ without any difficulty? The second problem, assuming we can overcome the first, is to show that perhaps the natural choice of kernel function

$$k := \sum_{n\geq 1} \overline{(A_{\overline{h}}\varphi_n)(\zeta)}\varphi_n$$

satisfies the interpolation condition in Corollary 2.7.

One could also ask whether or not one could extend our results to determine, as in Ahern-Clark, when the *derivatives* (of certain orders) of functions in $\operatorname{Rng} A_{\overline{h}}$, have non-tangential limits at $\zeta \in \mathbb{T}$.

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