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## 1 Introduction

The published origins of *uniform distribution* in the Brauer group can be traced to a definition which first appeared in a paper of Benard and Schacher [1] where it is shown that if the field extension $K$ over $\mathbf{Q}$ is abelian and $[A]$ is a class in the Schur subgroup $\mathcal{S}(K)$, then the fundamental identity

$$\mathrm{inv}_{\mathcal{P}}\ [A] \equiv b_\sigma\ \mathrm{inv}_{\mathcal{P}^\sigma}\ [A] \pmod 1$$

holds. Here $\mathcal{P}$ is any prime of $K$, $\sigma$ is any automorphism of $K$, and $\sigma(\epsilon_n) = (\epsilon_n)^{b_\sigma}$ where $n$ is the exponent of $[A]$, and $\epsilon_n$ is a primitive $n$-th root of unity in $K$. The intricacy of this formulation leads one to suspect that their "observation" lies at the core of number-theoretic studies of the Brauer group in general, and the Schur subgroup in particular. This is indeed the case. The underlying assumption that $K$ actually contains a primitive $n$-th root of unity was culled from the folklore by Janusz [8] as a precursor to his tour-de-force classification of the Schur subgroup over an algebraic number field. Janusz also established the rationale for studying the unusual action of the automorphism group on elements of the Brauer group that appears in the fundamental identity by using it to develop a neat criterion for extending central automorphisms of central simple algebras [9]. DeMeyer [2] has considered this action from a more general ring theoretic and homological standpoint.

Though the clearest mathematical exposition of the underlying ideas necessary for, and leading up to, the fundamental identity may be found in Yamada [11], the historical account therein is unfortunately muddled. The reason being that the principals — Schacher, Benard, Janusz, Fein, Ford, DeMeyer et al — were both in close proximity and closely cooperating at the time, and thus individual contributions from their differing perspectives (e.g., number theory, character theory, representation theory, ring theory, homological algebra, etc.) became blurred with the passage of time.

Renewed interest in uniform distribution began when Mollin freed the fundamental identity from its Schur subgroup ties [10]. Shortly thereafter the author provided additional momentum by circumventing the roots of unity condition [6]. But the truly surprising and exciting discoveries occurred when the fundamental identity was generalized from the Brauer group of a field to the Brauer group of a (commutative) ring [3, 4, 5, 7]. Now the action of the automorphism group became the driving force and assumed its rightful prominence.

In this progress/technical report our objective is twofold. First, to formalize and expand upon remarks appearing in [7] concerning the relativization of the fundamental identity in the setting of the Brauer group of a ring, and second to exhibit a construction which shows how to interpret uniform distribution as a homological phenomenon.

## 2 Relative Distribution

We assume throughout that the commutative ring $R$ is *connected* (viz., 0 and 1 are the only idempotents). Fix a map $c : \mathrm{Aut}(R) \longrightarrow \mathbf{Z}$. Recall that for $\sigma \in \mathrm{Aut}(R)$, the natural action of $\sigma$ on the Brauer group $\mathcal{B}(R)$ is given via $\sigma \circ [A] = [A_\sigma]$ where $A_\sigma = A$ as a ring, with $R$-module action given by $r \star a = \sigma^{-1}(r)a$.

*Definition 2.1* The *relative* uniform distribution group $\mathcal{R}_c(R)$ consists of those classes $[A] \in \mathcal{B}(R)$ such that $[A_\sigma] = [A]^{c(\sigma)}$, for all $\sigma \in \mathrm{Aut}(R)$.

*Remark.* Classically, $c$ is realized by taking the action of $\sigma$ on $< \epsilon_n > \cong \mathbf{Z}_\mathbf{n}^*$, where $n$ is *maximal* subject to $\epsilon_n \in R$.

*Proposition 2.2* $\mathcal{R}_c(R)$ is a subgroup of $\mathcal{B}(R)$.
*Proof.* Since $\sigma \circ (A \otimes B) = (\sigma \circ A) \otimes (\sigma \circ B)$, [2, Lemma 1], this is immediate.

The purpose of the following lemma is to show that $\mathcal{R}_c(-)$ respects the standard decomposition of $\mathcal{B}(-)$ into prime power indices.

*Lemma 2.3* If $[A], [B] \in \mathcal{B}(R)$ have exponents $m$, $n$ with $(m, n) = 1$ and $[A \otimes B] \in \mathcal{R}_c(R)$ then $[A], [B] \in \mathcal{R}_c(R)$.
*Proof.* By the previous proposition, $[A \otimes B]^n = [A]^n \in \mathcal{R}_c(R)$. Write $xm + yn = 1$. Then $[R]^x([A]^n)^y = ([A]^m)^x([A]^n)^y = [A]^{mx+ny} = [A] \in \mathcal{R}_c(R)$.

*Lemma 2.4* If $[A] \in \mathcal{R}_c(R)$ has exponent $m$ then $c_{\sigma\tau} \equiv c_\sigma c_\tau \pmod{m}$.
*Proof.* Since $\sigma\tau \circ [A] = \sigma \circ (\tau \circ [A]))$ we have $[A]^{c_{\sigma\tau}} = [A_{\sigma\tau}] = \sigma \circ [A_\tau] = \sigma \circ ([A]^{c_\tau}) = (\sigma \circ [A])^{c_\tau} = [A_\sigma]^{c_\tau} = [A]^{c_\sigma c_\tau}$.

*Corollary 2.5* If $[A] \in \mathcal{R}_c(R)$ has exponent $m$ then $(c_\sigma, m) = 1$ for all $\sigma$.

*Proof.* If $\imath$ is the identity, $[A] = [A_\imath] = [A]^{c(\imath)}$, yielding

$$1 \equiv c_\imath \equiv c_\sigma c_{\sigma^{-1}} \pmod{m}$$

so $c_\sigma$ is invertible modulo $m$.

*Theorem 2.6* If there exists $[A] \in \mathcal{R}_c(R)$ of exponent $m$ then

$$\overline{c} = \pi_m \circ c : \mathrm{Aut}(R) \longrightarrow \mathbf{Z}_\mathbf{m}^*$$

is a (group) homomorphism, where $\pi_m$ is the canonical projection.

*Proof.* The corollary shows that the map is well-defined, and the previous lemma that it is a homomorphism.

Of course, as in [6], one would like to prove a converse to the theorem. The difficulty lies not with the theory of uniform distribution, but with the structure theory of the Brauer group of a ring! Specifically, one needs to know how to construct elements of the Brauer group with proscribed automorphism action, and this seems hopeless at present unless one chooses to mimic the theory for fields by perhaps restricting oneself to the case where $\mathcal{B}(R) \longrightarrow \mathcal{B}(K)$ is a monomorphism, $K$ the quotient field of the *domain* $R$.

Clearly the results of this section are unchanged if we replace $\mathrm{Aut}(R)$ by one of its subgroups $G$. We denote this altered group by $\mathcal{R}_c(R, G)$ and, thanks to [6, Theorem 3.6], we are able to present our final result of this section.

*Proposition 2.7.* If $G$ has finite exponent and $\mathrm{Im}(c) \not\subset < -1 >$ then $\mathcal{R}_c(R, G)$ is of bounded exponent.

*Proof.* Let $n$ be the exponent of $G$ and set $M = 1 + \max\{|c_\sigma|^n\}$. By hypothesis, $n$ is finite and $M > 1$. Let $[A] \in \mathcal{R}_c(R, G)$ be of exponent $N > 1$. The calculation $[A] = [A_{\sigma^n}] = [A]^{(c_\sigma)^n}$ shows $N$ divides $(c_\sigma)^n$, whence $N < M$.

# 3 Equivalence

In this section we assume $K$ is a Galois extension of the *field* $F$, with Galois group $G$, and we consider the relationship between *two* classifying maps $c_1, c_2 : G \longrightarrow \mathbf{Z}$. To keep our context manageable we require that the assumptions necessary for Proposition 2.7 be met, so we assume throughout that $G$ is of finite exponent and $\mathrm{Im}(c_i) \not\subset < -1 >$. For ease of notation we write $\mathcal{R}_{c_i}$ for $\mathcal{R}_{c_i}(K, G)$.

*Theorem 3.1* $\mathcal{R}_{c_1} = \mathcal{R}_{c_2}$ if and only if $\pi_n \circ c_1 = \pi_n \circ c_2$ where $n$ is the (common) exponent of the $\mathcal{R}_{c_i}$.

*Proof.* The justification for the existence of such an $n$ is given above. Moreover $[A] \in \mathcal{R}_{c_1} \cap \mathcal{R}_{c_2}$ if and only if $[A]^{c_1(\sigma)} = [A_\sigma] = [A]^{c_2(\sigma)}$ which, in turn, occurs if and only if $c_1(\sigma) \equiv c_2(\sigma) \pmod{n}$.

Since any notion of *equivalence* for relative groups must preserve group exponents of the $\mathcal{R}_{c_i}$ we are led to consider (using $n$ for the common exponent) the diagram:

$$G \xrightarrow{c_i} \mathbf{Z} \xrightarrow{\pi_n} \mathbf{Z}_\mathbf{n}^*$$

which gives rise to the field diagram:

$$\begin{array}{c} K \\ | \\ E_i \\ | \\ F \end{array}$$

where $E_i$ is the fixed field of the kernel of the homonorphism $\overline{c_i}$. Since the $E_i$ are *normal* over $F$, it is not possible to define equivalence directly by requiring that the fields $E_i$ be $F$-isomorphic, but rather indirectly by requiring either that the groups $\mathrm{Gal}(E_i/F)$ be isomorphic or that the kernels of the $\overline{c_i}$ be isomorphic. But observe that because these kernels must contain the center of $G$ the associated factor groups must be abelian, so the conditions are necessarily the same.

*Definition 3.2* Let $\mathcal{R}_{c_1}, \mathcal{R}_{c_2}$ both have exponent $n$. We define $\mathcal{R}_{c_1} \sim \mathcal{R}_{c_2}$ if and only if $\ker(\overline{c_1}) \cong \ker(\overline{c_2})$.

From this definition it is easy to show that $\mathcal{R}_{c_1}$ is (group) isomorphic to $\mathcal{R}_{c_2}$. Moreover this explains why irritating examples like those appearing in [6, Section 3] are unavoidable. We are once again confronting the reality that the Brauer group is not distinguishable on the basis of isomorphism type alone.

# 4 Homological Considerations

We mandated at the outset use of the *natural* action of $\mathrm{Aut}(R)$ on $\mathcal{B}(R)$ given by $\sigma \circ [A] = [A_\sigma]$. The question which arises therefore is whether or not there are any other actions which might be considered! One way to spruce up the natural action is to reformulate it it terms of automorphisms of $\mathcal{B}(R)$. Formally then, we shall consider maps $\rho : \mathrm{Aut}(R) \longrightarrow \mathrm{Aut}(\mathcal{B}(R))$ which satisfy $\rho(\sigma)([A]) = [A_{\rho(\sigma)}]$. The natural map is obtained simply by taking $\rho$ to be the identity map.

We have seen previously that it is not $\mathrm{Aut}(R)$ that controls uniform distribution but, via the canonical projections, the image of the classifying map in $\mathbf{Z}^*_\mathbf{n}$, the units group of $\mathbf{Z}/n$. Since we would like to have a uniform distribution "acceptor" for any element in $\mathcal{B}(R)$, $n$ should be arbitrary, so in fact we need to consider maps:

$$\overset{\lim}{\leftarrow} \mathrm{units}(\mathbf{Z}/n) \xrightarrow{\gamma} \mathrm{Aut}(\mathcal{B}(R))$$

with the understanding that $\gamma(\bar{\jmath})([A]) = [A]^{\gamma(\bar{\jmath})}$. Thus homologically the uniform distribution group $\mathcal{U}_\mathcal{H}(R)$ may be defined as

$$\mathcal{U}_\mathcal{H}(R) = \{[A] \in \mathcal{B}(R) : \rho(\sigma) \circ [A] = \gamma(t(\sigma)) \circ [A], \text{ for all } \sigma\}$$

which is to say, using the maps shown in the following diagram ($G$ is used if *relativized* versions are desired) that the lower right triangle must commute!

$$\begin{array}{ccc}
G & \longrightarrow & \overset{\lim}{\leftarrow} \mathrm{units}(\mathbf{Z}/n) \\
| & & | \\
 & t & \\
\downarrow & & \downarrow \gamma \\
\mathrm{Aut}(R) & \xrightarrow{\rho} & \mathrm{Aut}(\mathcal{B}(R))
\end{array}$$

Though it is hardly surprising that we have reached the limits of our ability to analyze the structure of uniform distribution, there are three things to note about our construction.

1. Since an *inverse* limit is required in order to preserve the essence of the fundamental identity, the only natural action that *can* be used is the one we have specified. That is, as is well known, inverse limits are not functorial with respect to the Brauer group.

2. We can think of the above definition as parameterizing uniform distribution, in the sense that we can range from the simple case where $\bar{\jmath}$ is trivial, obtaining the *identically* distributed subgroups, all the way up to the relative maps $\bar{c}$ that we actually chose to consider.

3. The difficulty in defining any notion of equivalence is now seen as the obstruction to constructing the map $\rho$.

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