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C*-Algebras Generated by Truncated Toeplitz Operators

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C^*-algebras generated by truncated Toeplitz operators

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Dedicated to the memory of William Arveson.

Abstract. We obtain an analogue of Coburn's description of the Toeplitz algebra in the setting of truncated Toeplitz operators. As a byproduct, we provide several examples of complex symmetric operators which are not unitarily equivalent to truncated Toeplitz operators having continuous symbols.

1. Introduction

In the following, we let $\mathcal{H}$ denote a separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. For each $\mathscr{X} \subseteq \mathcal{B}(\mathcal{H})$, let $C^*(\mathscr{X})$ denote the unital C^*-algebra generated by $\mathscr{X}$. Since we are frequently interested in the case where $\mathscr{X} = \{A\}$ is a singleton, we often write $C^*(A)$ in place of $C^*(\{A\})$ in order to simplify our notation.

Recall that the *commutator ideal* $\mathscr{C}(C^*(\mathscr{X}))$ of $C^*(\mathscr{X})$ is the smallest norm closed two-sided ideal which contains the *commutators* $[A, B] := AB - BA$, where A and B range over all elements of $C^*(\mathscr{X})$. Since the quotient algebra

$$C^*(\mathscr{X})/\mathscr{C}(C^*(\mathscr{X}))$$

is an abelian C^*-algebra, it is isometrically $*$-isomorphic to $C(Y)$, the set of all continuous functions on some compact Hausdorff space Y [12, Thm. 1.2.1]. If we

agree to denote isometric $*$-isomorphism by $\cong$, then we may write

$$\frac{C^*(\mathscr{X})}{\mathscr{C}(C^*(\mathscr{X}))} \cong C(Y). \tag{1}$$

This yields the short exact sequence

$$0 \longrightarrow \mathscr{C}(C^*(\mathscr{X})) \stackrel{\iota}{\longrightarrow} C^*(\mathscr{X}) \stackrel{\pi}{\longrightarrow} C(Y) \longrightarrow 0, \tag{2}$$

where $\iota : \mathscr{C}(C^*(\mathscr{X})) \to C^*(\mathscr{X})$ is the inclusion map and $\pi : C^*(\mathscr{X}) \to C(Y)$ is the composition of the quotient map with the map which implements (1).

The *Toeplitz algebra* $C^*(T_z)$, where T_z denotes the unilateral shift on the classical Hardy space H^2, has been extensively studied since the seminal work of Coburn in the late 1960s [10, 11]. Indeed, the Toeplitz algebra is now one of the standard examples discussed in many well-known texts (e.g., [3, Sect. 4.3], [13, Ch. V.1], [14, Ch. 7]). In this setting, we have $\mathscr{C}(C^*(T_z)) = \mathscr{K}$, the ideal of compact operators on H^2, and $Y = \mathbb{T}$ (the unit circle), so that the short exact sequence (2) takes the form

$$0 \longrightarrow \mathscr{K} \stackrel{\iota}{\longrightarrow} C^*(T_z) \stackrel{\pi}{\longrightarrow} C(\mathbb{T}) \longrightarrow 0. \tag{3}$$

In other words, $C^*(T_z)$ is an *extension* of $\mathscr{K}$ by $C(\mathbb{T})$. In fact, one can prove that

$$C^*(T_z) = \{T_\varphi + K : \varphi \in C(\mathbb{T}), K \in \mathscr{K}\}$$

and that each element of $C^*(T_z)$ enjoys a unique decomposition of the form $T_\varphi + K$ [3, Thm. 4.3.2]. Indeed, it is well-known that the only compact Toeplitz operator is the zero operator [3, Cor. 1, p. 109]. We also note that the surjective map $\pi : C^*(T_z) \to C(\mathbb{T})$ in (3) is given by $\pi(T_\varphi + K) = \varphi$.

The preceding results have spawned numerous generalizations and variants over the years. For instance, one can consider C^*-algebras generated by matrix-valued Toeplitz operators or by Toeplitz operators which act upon other Hilbert function spaces (e.g., the Bergman space [4,25]). As another example, if $\mathscr{X}$ denotes the space of functions on $\mathbb{T}$ which are both piecewise and left continuous, then a fascinating result of Gohberg and Krupnik asserts that $\mathscr{C}(C^*(\mathscr{X})) = \mathscr{K}$ and provides the short exact sequence

$$0 \longrightarrow \mathscr{K} \stackrel{\iota}{\longrightarrow} C^*(\mathscr{X}) \stackrel{\pi}{\longrightarrow} C(Y) \longrightarrow 0,$$

where Y is the cylinder $\mathbb{T} \times [0, 1]$, endowed with a certain nonstandard topology [21].

Along different lines, we seek here to replace Toeplitz operators with *truncated Toeplitz operators*, a class of operators whose study has been largely motivated by a seminal 2007 paper of Sarason [26]. Let us briefly recall the basic definitions which are required for this endeavor. We refer the reader to Sarason's paper or to the recent survey article [18] for a more thorough introduction.

For each nonconstant inner function u, we consider the *model space*

$$\mathcal{K}_u := H^2 \ominus u H^2,$$

which is simply the orthogonal complement of the standard Beurling-type subspace uH^2 of H^2. Letting P_u denote the orthogonal projection from $L^2 := L^2(\mathbb{T})$ onto

$\mathcal{K}_u$, for each φ in $L^\infty(\mathbb{T})$ we define the *truncated Toeplitz operator* $A_\varphi^u : \mathcal{K}_u \to \mathcal{K}_u$ by setting

$$A_\varphi^u f = P_u(\varphi f)$$

for f in $\mathcal{K}_u$. The function φ in the preceding is referred to as the *symbol* of the operator A_φ^u.[1] In particular, let us observe that A_φ^u is simply the compression of the standard Toeplitz operator $T_\varphi : H^2 \to H^2$ to the subspace $\mathcal{K}_u$. Unlike traditional Toeplitz operators, however, the symbol of a truncated Toeplitz is not unique. In fact, $A_\varphi^u = 0$ if and only if φ belongs to $uH^2 + \overline{uH^2}$ [26, Thm. 3.1].

In our work, the *compressed shift* A_z^u plays a distinguished role analogous to that of the unilateral shift T_z in Coburn's theory. In light of this, let us recall that the spectrum $\sigma(A_z^u)$ of A_z^u coincides with the so-called *spectrum*

$$\sigma(u) := \left\{ \lambda \in \mathbb{D}^- : \liminf_{z \to \lambda} |u(z)| = 0 \right\} \tag{4}$$

of the inner function u [26, Lem. 2.5]. In particular, if $u = b_\Lambda s_\mu$, where b_Λ is a Blaschke product with zero sequence $\Lambda = \{\lambda_n\}$ and s_μ is a singular inner function with corresponding singular measure μ, then

$$\sigma(u) = \Lambda^- \cup \operatorname{supp}\mu.$$

With this terminology and notation in hand, we are ready to state our main result, which provides an analogue of Coburn's description of the Toeplitz algebra in the truncated Toeplitz setting.

Theorem 1. *If u is an inner function, then*

(i) $\mathscr{C}(C^*(A_z^u)) = \mathscr{K}$, *the algebra of compact operators on $\mathcal{K}_u$,*

(ii) $C^*(A_z^u)/\mathscr{K}$ *is isometrically $*$-isomorphic to $C(\sigma(u) \cap \mathbb{T})$,*

(iii) *For φ in $C(\mathbb{T})$, A_φ^u is compact if and only if $\varphi(\sigma(u) \cap \mathbb{T}) = \{0\}$,*

(iv) $C^*(A_z^u) = \{A_\varphi^u + K : \varphi \in C(\mathbb{T}), K \in \mathscr{K}\}$,

(v) *For φ in $C(\mathbb{T})$, $\sigma_e(A_\varphi^u) = \varphi(\sigma_e(A_z^u))$,*

(vi) *For φ in $C(\mathbb{T})$, $\|A_\varphi^u\|_e = \sup\{|\varphi(\zeta)| : \zeta \in \sigma(u) \cap \mathbb{T}\}$,*

(vii) *Every operator in $C^*(A_z^u)$ is of the form normal plus compact.*

Moreover,

$$0 \longrightarrow \mathscr{K} \xrightarrow{\iota} C^*(A_z^u) \xrightarrow{\pi} C(\sigma(u) \cap \mathbb{T}) \longrightarrow 0$$

is a short exact sequence and thus $C^*(A_z^u)$ is an extension of the compact operators by $C(\sigma(u) \cap \mathbb{T})$. In particular, the map $\pi : C^*(A_z^u) \to C(\sigma(u) \cap \mathbb{T})$ is given by

$$\pi(A_\varphi^u + K) = \varphi|_{\sigma(u) \cap \mathbb{T}}.$$

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[1] It is possible to consider truncated Toeplitz operators with symbols in $L^2(\mathbb{T})$, although we have little need to do so here.

The proof of Theorem 1 is somewhat involved and requires a number of preliminary lemmas. It is therefore deferred until Section 3. However, let us remark now that the same result holds when the hypothesis that f belongs to $C(\mathbb{T})$ is replaced by the weaker assumption that f is in $\mathscr{X}_u$, the class of L^∞ functions which are continuous at each point of $\sigma(u) \cap \mathbb{T}$. In fact, given f in $\mathscr{X}_u$, there exists a g in $C(\mathbb{T})$ so that

$$A_f^u \equiv A_g^u \pmod{\mathscr{K}}.$$

Thus one can replace A_f^u with A_g^u when working modulo the compact operators and adapt the proof of Theorem 1 so that $C(\mathbb{T})$ is replaced by $\mathscr{X}_u$.

Using completely different language and terminology, some aspects of Theorem 1 can be proven by triangularizing the compressed shift A_z^u according to the scheme discussed at length in [24, Lec. V]. For instance, items (vi) and (iii) of the preceding theorem are [1, Cor. 5.1] and [1, Thm. 5.4], respectively (we should also mention related work of Kriete [22, 23]). From an operator algebraic perspective, however, we believe that a different approach is desirable. Our approach is similar in spirit to the original work of Coburn and forms a possible blueprint for variations and extensions (see Section 4). Moreover, our approach does not require the detailed consideration of several special cases (i.e., Blaschke products, singular inner functions with purely atomic spectra, etc.) as does the approach pioneered in [1, 2]. In particular, we are able to avoid the somewhat involved computations and integral transforms encountered in the preceding references.

2. Continuous symbols and the TTO-CSO problem

Recall that a bounded operator T on a Hilbert space $\mathcal{H}$ is called *complex symmetric* if there exists a conjugate-linear, isometric involution J on $\mathcal{H}$ such that $T = JT^*J$. It was first recognized in [16, Prop. 3] that every truncated Toeplitz operator is complex symmetric (see also [15] where this is discussed in great detail). This hidden symmetry turns out to be a crucial ingredient in Sarason's general treatment of truncated Toeplitz operators [26].

A significant amount of evidence is mounting that truncated Toeplitz operators may play a significant role in some sort of model theory for complex symmetric operators. Indeed, a surprising and diverse array of complex symmetric operators can be concretely realized in terms of truncated Toeplitz operators (or direct sums of such operators). The recent articles [7, 8, 19, 27] all deal with various aspects of this problem and a survey of this work can be found in [18, Sect. 9].

It turns out that viewing truncated Toeplitz operators in the C^*-algebraic setting can shed some light on the question of whether every complex symmetric operator can be written in terms of truncated Toeplitz operators (the *TTO-CSO Problem*). Corollaries 2 and 3 below provide examples of complex symmetric operators which are not unitarily equivalent to truncated Toeplitz operators having continuous symbols. To our knowledge, this is the first *negative* evidence relevant to the TTO-CSO Problem which has been obtained.

Corollary 2. *If A is a noncompact operator on a Hilbert space $\mathcal{H}$, then the operator $T : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ defined by*

$$T = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

is a complex symmetric operator which is not unitarily equivalent to a truncated Toeplitz operator with continuous symbol.

Proof. Since T is nilpotent of degree two, it is complex symmetric by [20, Thm. 2]. However, T is not of the form normal plus compact since $[T, T^*]$ is noncompact. Thus T cannot belong to $C^*(A_z^u)$ for any u by (vii) of Theorem 1. $\quad\square$

Corollary 3. *If S denotes the unilateral shift, then $T = \bigoplus_{i=1}^{\infty}(S \oplus S^*)$ is a complex symmetric operator which is not unitarily equivalent to a truncated Toeplitz operator with continuous symbol.*

Proof. First note that the operator $S \oplus S^*$ is complex symmetric by [17, Ex. 5] whence T itself is complex symmetric. Since $[S, S^*]$ has rank one, it follows that $[T, T^*]$ is noncompact. Therefore T is not of the form normal plus compact whence T cannot belong to $C^*(A_z^u)$ for any u by (vii) of Theorem 1. $\quad\square$

Unfortunately, the preceding corollary sheds no light on the following apparently simple problem.

Problem 1. Is $S \oplus S^*$ unitarily equivalent to a truncated Toeplitz operator? If so, can the symbol be chosen to be continuous?

3. Proof of Theorem 1

To prove Theorem 1, we first require a few preliminary lemmas. The first lemma is well-known and we refer the reader to [24, p. 65] or [6, p. 84] for its proof.

Lemma 1. *Each function in $\mathcal{K}_u$ can be analytically continued across $\mathbb{T} \backslash \sigma(u)$.*

The following description of the spectrum and essential spectrum of the compressed shift can be found in [26, Lem. 2.5], although portions of it date back to the work of Livšic and Moeller [24, Lec. III.1]. The essential spectrum of A_z^u was computed in [1, Cor. 5.1].

Lemma 2. $\sigma(A_z^u) = \sigma(u)$ *and* $\sigma_e(A_z^u) = \sigma(u) \cap \mathbb{T}$.

Although the following must certainly be well-known among specialists, we do not recall having seen its proof before in print. We therefore provide a short proof of this important fact.

Lemma 3. A_z^u *is irreducible.*

Proof. Let $\mathcal{M}$ be a nonzero reducing subspace of $\mathcal{K}_u$ for the operator A_z^u. In light of the fact that $\mathcal{M}$ is invariant under the operator $I - A_z^u(A_z^u)^* = k_0 \otimes k_0$ [26, Lem. 2.4], it follows that the nonzero vector k_0 belongs to $\mathcal{M}$. Since k_0 is a cyclic vector for A_z^u [26, Lem. 2.3], we conclude that $\mathcal{M} = \mathcal{K}_u$. $\qquad\square$

Lemma 4. *If $\varphi \in C(\mathbb{T})$, then A_φ^u is compact if and only if $\varphi|_{\sigma(u) \cap \mathbb{T}} \equiv 0$.*

Proof. ($\Leftarrow$) Suppose that $\varphi|_{\sigma(u) \cap \mathbb{T}} \equiv 0$. Let $\varepsilon > 0$ and pick ψ in $C(\mathbb{T})$ such that ψ vanishes on an open set containing $\sigma(u) \cap \mathbb{T}$ and $\|\varphi - \psi\|_\infty < \varepsilon$. Since $\|A_\varphi^u - A_\psi^u\| \le \|\varphi - \psi\|_\infty < \varepsilon$, it suffices to show that A_ψ^u is compact. To this end, we prove that if f_n is a sequence in $\mathcal{K}_u$ which tends weakly to zero, then $A_\psi^u f_n \to 0$ in norm.

Let K denote the closure of $\psi^{-1}(\mathbb{C}\backslash\{0\})$ and note that $K \subset \mathbb{T} \backslash \sigma(u)$. By Lemma 1, we know that each f_n has an analytic continuation across K from which it follows that $f_n(\zeta) = \langle f_n, k_\zeta \rangle \to 0$, where

$$k_\zeta(z) = \frac{1 - \overline{u(\zeta)}u(z)}{1 - \overline{\zeta}z}$$

denotes the reproducing kernel corresponding to a point ζ in K [26, p. 495]. Since u is analytic on a neighborhood of the compact set K we obtain

$$|f_n(\zeta)| = |\langle f_n, k_\zeta \rangle| \le \|f_n\| |u'(\zeta)|^{\frac{1}{2}} \le \sup_n \|f_n\| \sup_{\zeta \in K} |u'(\zeta)|^{\frac{1}{2}} = C < \infty$$

for each ζ in K. By the dominated convergence theorem, it follows that

$$\|A_\psi^u f_n\|^2 = \|P_u(\psi f_n)\| \le \|\psi f_n\|^2 = \int_K |\psi|^2 |f_n|^2 \to 0$$

whence $A_\psi^u f_n$ tends to zero in norm, as desired.

($\Rightarrow$) Suppose that φ belongs to $C(\mathbb{T})$, ξ belongs to $\sigma(u) \cap \mathbb{T}$, and A_φ^u is compact. Let

$$F_\lambda(z) := \frac{1 - |\lambda|^2}{1 - |u(\lambda)|^2} \left| \frac{1 - \overline{u(\lambda)}u(z)}{1 - \overline{\lambda}z} \right|^2,$$

which is the absolute value of the normalized reproducing kernel for $\mathcal{K}_u$. Observe that $F_\lambda(z) \ge 0$ and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F_\lambda(e^{it}) \, dt = 1$$

by definition.

By (4) there is sequence λ_n in $\mathbb{D}$ such that $|u(\lambda_n)| \to 0$. Suppose that $\xi = e^{i\alpha}$ and note that if $|t - \alpha| \ge \delta$, then

$$F_{\lambda_n}(e^{it}) \le C_\delta \frac{1 - |\lambda_n|^2}{1 - |u(\lambda_n)|^2} \to 0. \tag{5}$$

This is enough to make the following approximate identity argument go through. Indeed,

$$\left| \varphi(\xi) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(e^{it}) F_{\lambda_n}(e^{it}) \, dt \right|$$

$$\leq \frac{1}{2\pi} \int_{|t-\alpha| \leq \delta} |\varphi(\xi) - \varphi(e^{it})| F_{\lambda_n}(e^{it})\, dt$$

$$+ \frac{1}{2\pi} \int_{|t-\alpha| \geq \delta} |\varphi(\xi) - \varphi(e^{it})| F_{\lambda_n}(e^{it})\, dt.$$

This first integral can be made small by the continuity of φ. Once $\delta > 0$ is fixed, the second term goes to zero by (5). $\qquad \square$

Remark 1. We would like to thank the referee for suggesting this elegant normalized kernel function proof of the $(\Rightarrow)$ direction of this lemma. Our original argument was somewhat longer.

Lemma 5. *For each $\varphi, \psi \in C(\mathbb{T})$, the semicommutator $A_\varphi^u A_\psi^u - A_{\varphi\psi}^u$ is compact. In particular, the commutator $[A_\varphi^u, A_\psi^u]$ is compact.*

Proof. Let $p(z) = \sum_i p_i z^i$ and $q(z) = \sum_j q_j z^j$ be trigonometric polynomials on $\mathbb{T}$ and note that

$$A_p^u A_q^u - A_{pq}^u = \sum_{i,j} p_i q_j (A_{z^i}^u A_{z^j}^u - A_{z^{i+j}}^u).$$

We claim that the preceding operator is compact. Since all sums involved are finite, it suffices to prove that $A_{z^i}^u A_{z^j}^u - A_{z^{i+j}}^u$ is compact for each pair (i,j) of integers.

If i and j are of the same sign, then $A_{z^i}^u A_{z^j}^u - A_{z^{i+j}}^u = 0$ is trivially compact. If i and j are of different signs, then upon relabeling and taking adjoints, if necessary, it suffices to show that if $n \geq m \geq 0$, then the operator $A_{z^n}^u A_{\bar{z}^m}^u - A_{z^{n-m}}^u$ is compact (the case $n \leq m \leq 0$ being similar). In light of the fact that

$$A_{z^n}^u A_{\bar{z}^m}^u - A_{z^{n-m}}^u = A_{z^{n-m}}^u (A_{z^m}^u A_{\bar{z}^m}^u - I),$$

we need only show that $A_{z^m}^u A_{\bar{z}^m}^u - I$ is compact for each $m \geq 1$. However, since $A_z^u A_{\bar{z}}^u - I$ has rank one [26, Lem. 2.4], this follows immediately from the identity

$$A_{z^m}^u A_{\bar{z}^m}^u - I = \sum_{\ell=0}^{m-1} A_{z^\ell}^u (A_z^u A_{\bar{z}}^u - I) A_{\bar{z}^\ell}^u.$$

Having shown that $A_p^u A_q^u - A_{pq}^u$ is compact for every pair of trigonometric polynomials p and q, the desired result follows since we may uniformly approximate any given φ, ψ in $C(\mathbb{T})$ by their respective Cesàro means. $\qquad \square$

Remark 2. For Toeplitz operators, it is known that the semicommutator $T_\varphi T_\psi - T_{\varphi\psi}$ is compact under the assumption that one of the symbols is continuous, while the other belongs to L^∞ [3, Prop. 4.3.1], [13, Cor. V.1.4]. Though not needed for the proof of our main theorem, the same is true for truncated Toeplitz operators. This was kindly pointed out to us by Trieu Le. Here is his proof: For f in L^∞, define the Hankel operator $H_f^u : \mathcal{K}_u \to L^2$ by $H_f^u := (I - P_u)M_f$ and note that $(H_f^u)^* = P_u M_{\overline{f}}(I - P_u)$. For φ, ψ in L^∞ a computation shows that

$$A_{\varphi\psi}^u - A_\varphi^u A_\psi^u = (H_{\overline{\varphi}}^u)^* H_\psi^u. \tag{6}$$

If f belongs to L^∞, then setting $\varphi = \overline{f}$ and $\varphi = f$, we have

$$(H_f^u)^* H_f^u = A_{\overline{f}f}^u - A_{\overline{f}}^u A_f^u.$$

For continuous f it follows from the previous Lemma that $(H_f^u)^* H_f^u$ and hence H_f^u is compact whenever f is continuous. From (6) we see that if one of φ or ψ is continuous then $A_{\varphi\psi}^u - A_\varphi^u A_\psi^u$ is compact.

Proof of Theorem 1. Before proceeding further, let us remark that statement (iii) has already been proven (see Lemma 4). We first claim that

$$C^*(A_z^u) = C^*(\{A_\varphi^u : \varphi \in C(\mathbb{T})\}), \tag{7}$$

noting that the containment $\subseteq$ in the preceding holds trivially. Since $(A_z^u)^* = A_{\overline{z}}^u$, it follows that A_p^u belongs to $C^*(A_z^u)$ for any trigonometric polynomial p. We may then uniformly approximate any given φ in $C(\mathbb{T})$ by its Cesàro means to see that A_φ^u belongs to $C^*(A_z^u)$. This establishes the containment $\supseteq$ in (7).

We next prove statement (i) of Theorem 1, which states that the commutator ideal $\mathscr{C}(C^*(A_z^u))$ of $C^*(A_z^u)$ is precisely $\mathscr{K}$, the set of all compact operators on the model space $\mathcal{K}_u$:

$$\mathscr{C}(C^*(A_z^u)) = \mathscr{K}. \tag{8}$$

The containment $\mathscr{C}(C^*(A_z^u)) \subseteq \mathscr{K}$ follows easily from (7) and Lemma 5. On the other hand, Lemma 3 tells us that A_z^u is irreducible, whence the algebra $C^*(A_z^u)$ itself is irreducible. Since $[A_z^u, A_{\overline{z}}^u] \neq 0$ is compact, it follows that $C^*(A_z^u) \cap \mathscr{K} \neq \{0\}$. By [12, Cor. 3.16.8], we conclude that $\mathscr{K} \subseteq \mathscr{C}(C^*(A_z^u))$, which establishes (8).

We now claim that

$$C^*(A_z^u) = \{A_\varphi^u + K : \varphi \in C(\mathbb{T}), K \in \mathscr{K}\}, \tag{9}$$

which is statement (iv) of Theorem 1. The containment $\subseteq$ in the preceding holds because the right-hand side of (9) is a C^*-algebra which contains A_z^u (mimic the first portion of the proof of [3, Thm. 4.3.2] to see this). On the other hand, the containment $\supseteq$ in (9) follows because $C^*(A_z^u)$ contains $\mathscr{K}$ by (8) and contains every operator of the form A_φ^u with φ in $C(\mathbb{T})$ by (7).

The map $\gamma : C(\mathbb{T}) \to C^*(A_z^u)/\mathscr{K}$ defined by

$$\gamma(\varphi) = A_\varphi^u + \mathscr{K}$$

is a homomorphism by Lemma 5 and hence $\gamma(C(\mathbb{T}))$ is a dense subalgebra of $C^*(A_z^u)/\mathscr{K}$ by (7). In light of Lemma 4, we see that

$$\ker \gamma = \{\varphi \in C(\mathbb{T}) : \varphi|_{\sigma(u)\cap\mathbb{T}} \equiv 0\}, \tag{10}$$

whence the map

$$\widetilde{\gamma} : C(\mathbb{T})/\ker\gamma \to C^*(A_z^u)/\mathscr{K} \tag{11}$$

defined by

$$\widetilde{\gamma}(\varphi + \ker\gamma) = A_\varphi^u + \mathscr{K}$$

is an injective $*$-homomorphism. By [13, Thm. I.5.5], it follows that $\widetilde{\gamma}$ is an isometric $*$-isomorphism. Since

$$C(\mathbb{T})/\ker\gamma \cong C(\sigma(u)\cap\mathbb{T}) \tag{12}$$

by (10), it follows that

$$\sigma_e(A_\varphi^u) = \sigma_{C(\sigma(u)\cap\mathbb{T})}(\varphi) = \varphi(\sigma(u)\cap\mathbb{T}) = \varphi(\sigma_e(A_z^u)),$$

where $\sigma_{C(\sigma(u)\cap\mathbb{T})}(\varphi)$ denotes the spectrum of φ as an element of the Banach algebra $C(\sigma(u)\cap\mathbb{T})$. This yields statement (v). We also note that putting (11) and (12) together shows that $C^*(A_z^u)/\mathscr{K}$ is isometrically $*$-isomorphic to $C(\sigma(u)\cap\mathbb{T})$, which is statement (ii).

We now need only justify statement (vii). To this end, recall that a seminal result of Clark [9] asserts that for each α in $\mathbb{T}$, the operator

$$U_\alpha := A_z^u + \frac{\alpha}{1 - \overline{u(0)}\alpha} k_0 \otimes C k_0 \tag{13}$$

on $\mathcal{K}_u$ is a cyclic unitary operator and, moreover, that every unitary, rank-one perturbation of A_z^u is of the form (13). A complete exposition of this important result can be found in the text [5]. Since

$$U_\alpha \equiv A_z^u \pmod{\mathscr{K}},$$

it follows that

$$\varphi(U_\alpha) \equiv A_\varphi^u \pmod{\mathscr{K}} \tag{14}$$

for every φ in $C(\mathbb{T})$. This is because the norm on $\mathcal{B}(\mathcal{K}_u)$ dominates the quotient norm on $\mathcal{B}(\mathcal{K}_u)/\mathscr{K}$ and since any φ in $C(\mathbb{T})$ can be uniformly approximated by trigonometric polynomials. Since $\mathscr{K} \subseteq C^*(A_z^u)$, it follows that

$$C^*(U_\alpha) + \mathscr{K} = C^*(A_z^u),$$

which yields the desired result. $\qquad\qquad\square$

4. Piecewise continuous symbols

Having obtained a truncated Toeplitz analogue of Coburn's work, it is of interest to see if one can also obtain a truncated Toeplitz version of Gohberg and Krupnik's results concerning Toeplitz operators with piecewise continuous symbols [21]. Although we have not yet been able to complete this work, we have obtained a few partial results which are worth mentioning.

Let $PC := PC(\mathbb{T})$ denote the $*$-algebra of piecewise continuous functions on $\mathbb{T}$. To get started, we make the simplifying assumption that u is inner and that

$$\sigma(u)\cap\mathbb{T} = \{1\}.$$

For instance, u could be a singular inner function with a single atom at 1 or a Blaschke product whose zeros accumulate only at 1. Let

$$\mathscr{A}_{PC}^u = \{A_\varphi^u : \varphi \in PC\}$$

denote the set of all truncated Toeplitz operators on $\mathcal{K}_u$ having symbols in PC. The following lemma identifies the commutator ideal of $C^*(\mathscr{A}^u_{PC})$.

Lemma 6. $\mathscr{C}(C^*(\mathscr{A}^u_{PC})) = \mathscr{K}$.

Proof. Let

$$\chi(e^{i\theta}) := 1 - \frac{\theta}{2\pi}, \qquad 0 \le \theta < 2\pi, \tag{15}$$

and notice that χ belongs to PC and satisfies

$$\chi_+(1) := \lim_{\theta \to 0} \chi(e^{i\theta}) = 1, \qquad \chi_-(1) := \lim_{\theta \to 2\pi} \chi(e^{i\theta}) = 0.$$

If φ is any function in PC, then it follows that

$$\varphi - \varphi_+(1)\chi - \varphi_-(1)(1 - \chi)$$

is continuous at 1 and assumes the value zero there. By the remarks following Theorem 1 in the introduction, we see that

$$A^u_\varphi \equiv \alpha A^u_\chi + \beta I \pmod{\mathscr{K}}, \tag{16}$$

where $\alpha = \varphi_+(1) - \varphi_-(1)$ and $\beta = \varphi_-(1)$. In light of (16) it follows that

$$[A_\varphi, A_\psi] \equiv 0 \pmod{\mathscr{K}}$$

for any φ, ψ in PC whence $\mathscr{C}(C^*(\mathscr{A}^u_{PC})) \subseteq \mathscr{K}$. Since A^u_z belongs to $\mathscr{A}^u_{PC}$, we conclude that $\mathscr{C}(C^*(\mathscr{A}^u_{PC}))$ contains the nonzero commutator $[A^u_z, A^u_{\bar{z}}]$ whence $C^*(\mathscr{A}^u_{PC})$ is irreducible by Lemma 3. Moreover, By [12, Cor. 3.16.8] we conclude that $\mathscr{K} \subseteq \mathscr{C}(C^*(\mathscr{A}^u_{PC}))$ which concludes the proof. $\square$

Lemma 7. $C^*(\mathscr{A}^u_{PC}) = C^*(A^u_\chi) + \mathscr{K}$.

Proof. The containment $\supseteq$ is clear from (16) since $C^*(\mathscr{A}^u_{PC})$ contains $\mathscr{K}$. Conversely, the containment $\subseteq$ follows immediately from (16). $\square$

From the discussion above and [13, Cor. I.5.6] we know that

$$\frac{C^*(\mathscr{A}^u_{PC})}{\mathscr{C}(C^*(\mathscr{A}^u_{PC}))} = \frac{C^*(A^u_\chi) + \mathscr{K}}{\mathscr{K}} \cong \frac{C^*(A^u_\chi)}{C^*(A^u_\chi) \cap \mathscr{K}}.$$

is a commutative C^*-algebra. Unfortunately, we are unable to identify the algebra $C^*(A^u_\chi)$ in a more concrete manner. This highlights the important fact that truncated Toeplitz operators such as A^u_χ, whose symbols are neither analytic nor coanalytic, are difficult to deal with.

Problem 2. Suppose that $\sigma(u) = \{1\}$. Give a concrete description of $C^*(A^u_\chi)$ where χ denotes the piecewise continuous function (15).

Problem 3. Provide an analogue of the Gohberg-Krupnik result for $\mathscr{A}^u_{PC}$. In other words, give a description of $C^*(\mathscr{A}^u_{PC})$ analogous to that of Theorem 1.

References

1. P. R. Ahern and D. N. Clark, *On functions orthogonal to invariant subspaces*, Acta Math. **124** (1970), 191–204.

2. ———, *On star-invariant subspaces*, Bull. Amer. Math. Soc. **76** (1970), 629–632.

3. William Arveson, *A short course on spectral theory*, Graduate Texts in Mathematics, vol. 209, Springer-Verlag, New York, 2002.

4. S. Axler, J. B. Conway, and G. McDonald, *Toeplitz operators on Bergman spaces*, Canad. J. Math. **34** (1982), no. 2, 466–483.

5. J. A. Cima, A. L. Matheson, and W. T. Ross, *The Cauchy transform*, Mathematical Surveys and Monographs, vol. 125, American Mathematical Society, Providence, RI, 2006.

6. J. A. Cima and W. T. Ross, *The backward shift on the Hardy space*, Mathematical Surveys and Monographs, vol. 79, American Mathematical Society, Providence, RI, 2000.

7. J. A. Cima, W. T. Ross, and W. R. Wogen, *Truncated Toeplitz operators on finite dimensional spaces*, Oper. Matrices **2** (2008), no. 3, 357–369.

8. Joseph A. Cima, Stephan Ramon Garcia, William T. Ross, and Warren R. Wogen, *Truncated Toeplitz operators: spatial isomorphism, unitary equivalence, and similarity*, Indiana Univ. Math. J. **59** (2010), no. 2, 595–620.

9. D. N. Clark, *One dimensional perturbations of restricted shifts*, J. Analyse Math. **25** (1972), 169–191.

10. L. A. Coburn, *The C^*-algebra generated by an isometry*, Bull. Amer. Math. Soc. **73** (1967), 722–726.

11. ———, *The C^*-algebra generated by an isometry. II*, Trans. Amer. Math. Soc. **137** (1969), 211–217.

12. J. B. Conway, *A course in operator theory*, Graduate Studies in Mathematics, vol. 21, American Mathematical Society, Providence, RI, 2000.

13. Kenneth R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996.

14. Ronald G. Douglas, *Banach algebra techniques in operator theory*, second ed., Graduate Texts in Mathematics, vol. 179, Springer-Verlag, New York, 1998.

15. S. R. Garcia, *Conjugation and Clark operators*, Recent advances in operator-related function theory, Contemp. Math., vol. 393, Amer. Math. Soc., Providence, RI, 2006, pp. 67–111.

16. S. R. Garcia and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1285–1315 (electronic).

17. ———, *Complex symmetric operators and applications. II*, Trans. Amer. Math. Soc. **359** (2007), no. 8, 3913–3931 (electronic).

18. S. R. Garcia and W. T. Ross, *Recent progress on truncated Toeplitz operators*, Fields Institute Proceedings (to appear), `http://arxiv.org/abs/1108.1858`.

19. Stephan Ramon Garcia, Daniel E. Poore, and William T. Ross, *Unitary equivalence to a truncated Toeplitz operator: analytic symbols*, Proc. Amer. Math. Soc. **140** (2012), no. 4, 1281–1295.

20. Stephan Ramon Garcia and Warren R. Wogen, *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc. **362** (2010), no. 11, 6065–6077.

21. I. C. Gohberg and N. Ja. Krupnik, *The algebra generated by the Toeplitz matrices*, Funkcional. Anal. i Priložen. **3** (1969), no. 2, 46–56.

22. T. L. Kriete, III, *A generalized Paley-Wiener theorem*, J. Math. Anal. Appl. **36** (1971), 529–555.

23. Thomas L. Kriete, *Fourier transforms and chains of inner functions*, Duke Math. J. **40** (1973), 131–143.

24. N. Nikolski, *Treatise on the shift operator*, Springer-Verlag, Berlin, 1986.

25. W. T. Ross, *Analytic continuation in Bergman spaces and the compression of certain Toeplitz operators*, Indiana Univ. Math. J. **40** (1991), no. 4, 1363–1386.

26. D. Sarason, *Algebraic properties of truncated Toeplitz operators*, Oper. Matrices **1** (2007), no. 4, 491–526.

27. E. Strouse, D. Timotin, and M. Zarrabi, *Unitary equivalence to truncated Toeplitz operators*, `http://arxiv.org/abs/1011.6055`.

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